CSC 4356
Interactive Computer Graphics
Lecture 3: Geometric Transformations (2D)

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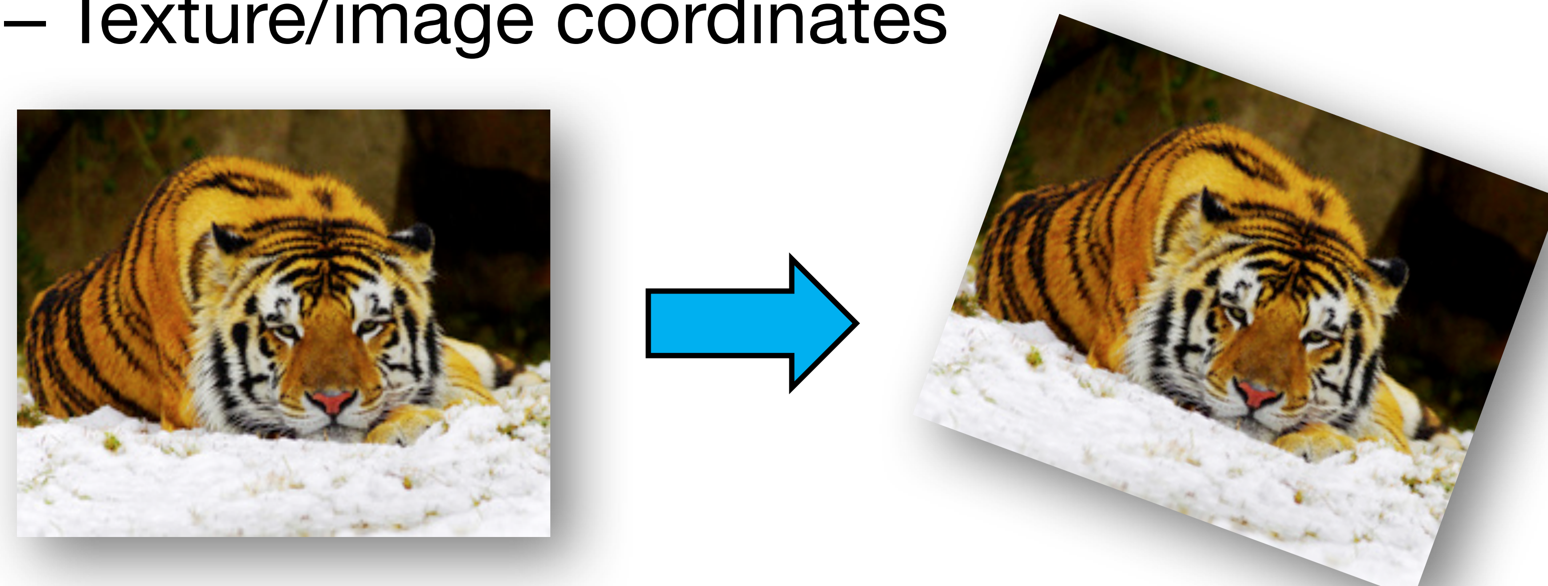
Tue & Thu: 10:30 - 11:50am
218 Tureaud Hall

Lecture 3: Geometric Transformations (2D)

- 2-Dimensional Transformations
  - Translation, Scaling, Rotation …
  - Inverse transformation
  - Composite transformation
- Homogeneous Coordinate
- Reading:
  - Textbook Chap 7

Motivation

- What is transformation?
  - Mapping points from one place to another
- What are 2D transformations for?
  - 2D primitives: lines, triangle, squares, etc.
  - Texture/image coordinates

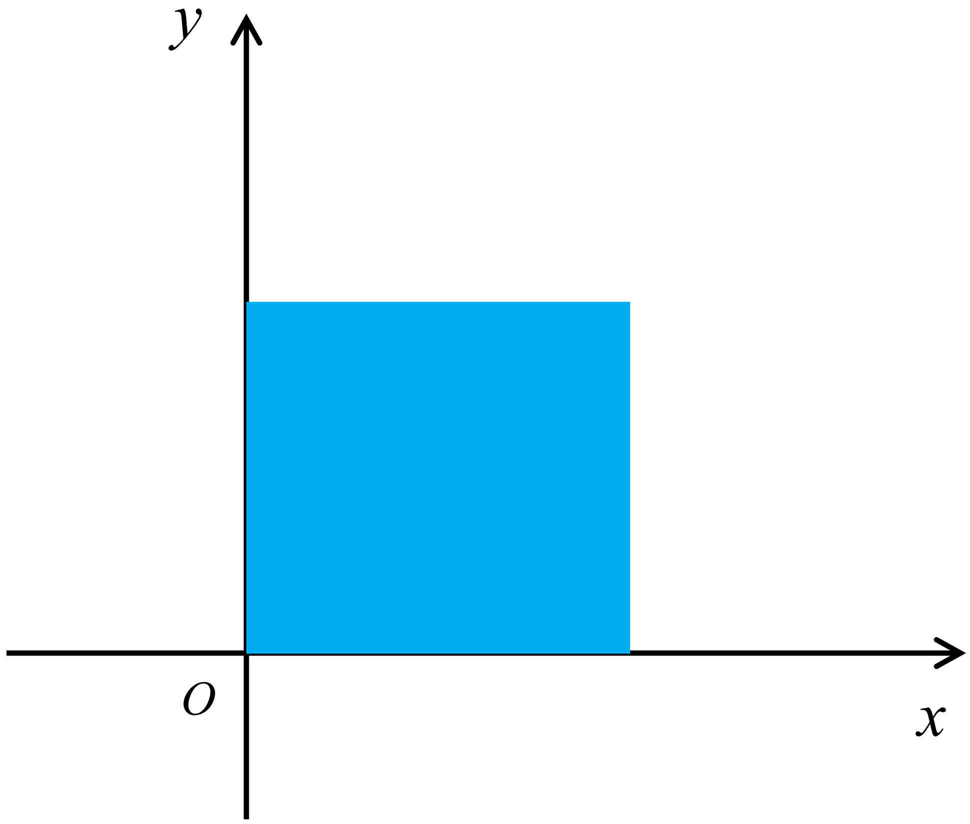


2D Transformations

- How to represent?
  - Matrix and vector operations (addition & multiplication)
- Basic 2D Transformations
  - Translation
  - Scaling
  - Rotation
  - Shear
  - Reflection

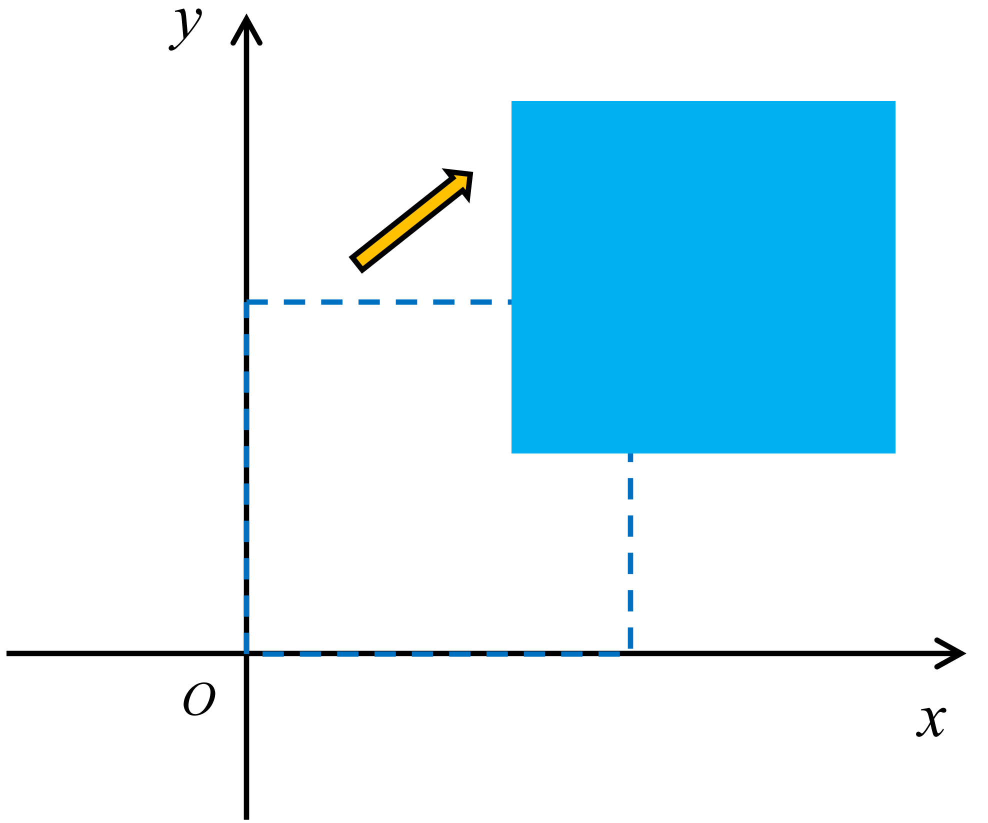
2D Translation

- Move object from one location to another



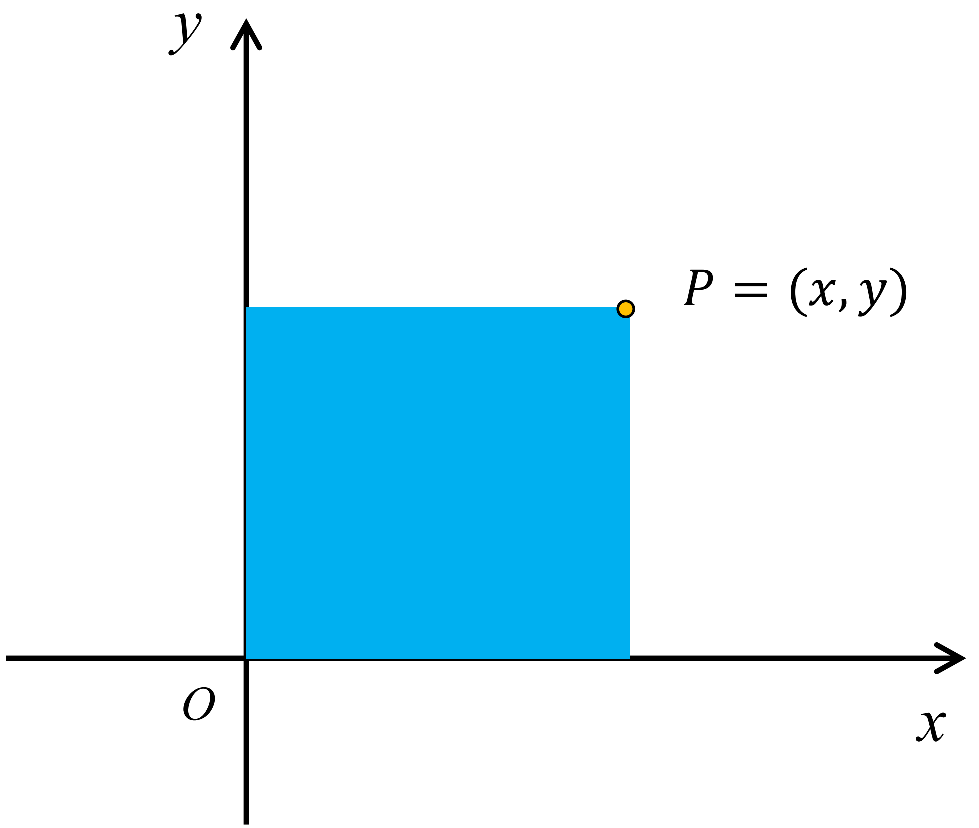
2D Translation

- Move object from one location to another



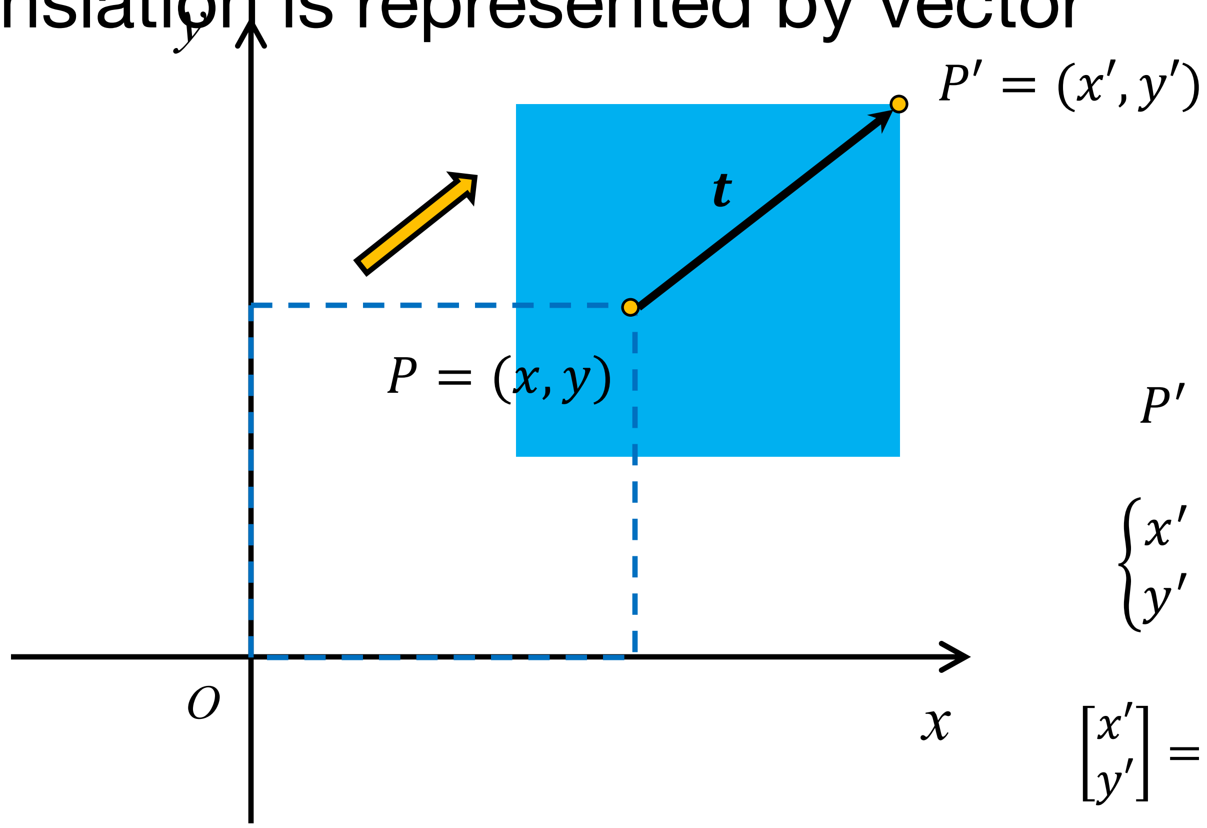
2D Translation

- Move object from one location to another



2D Translation

- Move object from one location to another
  - Translation is represented by vector



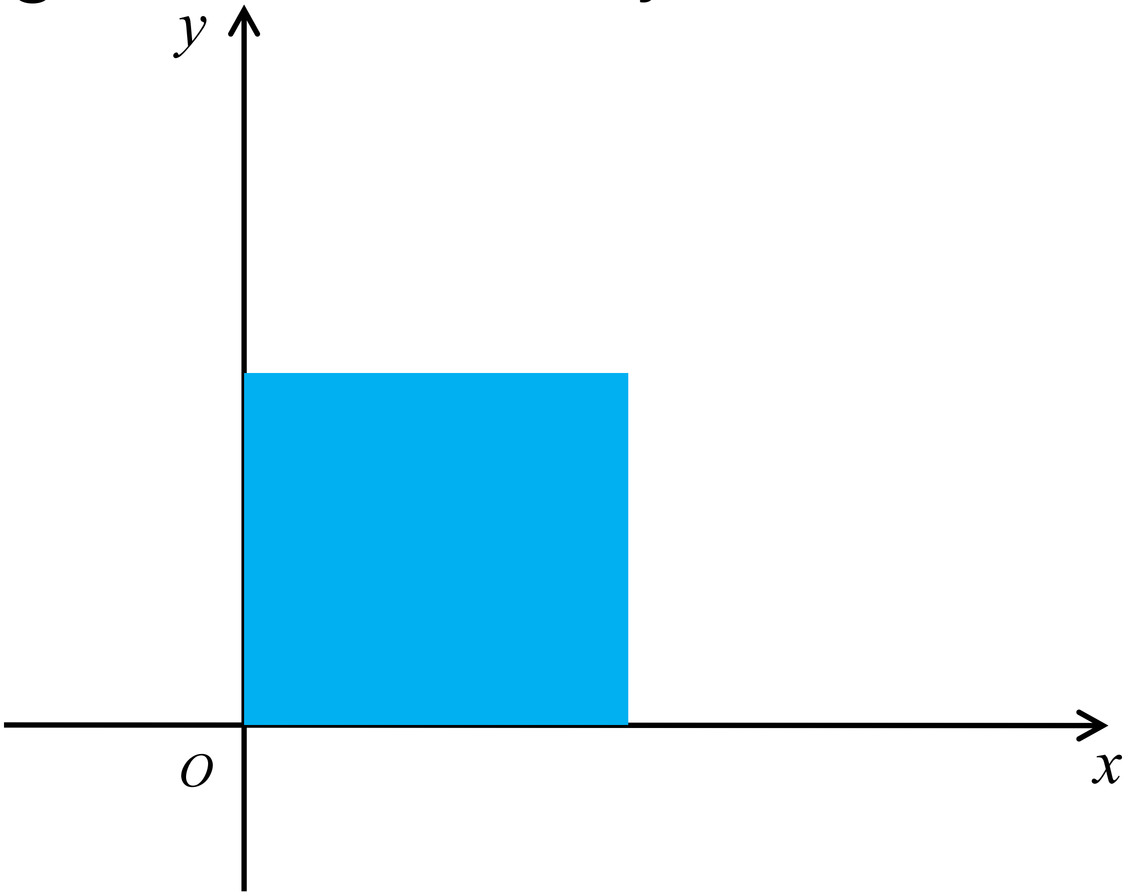
$$P' = P + \boldsymbol{t}$$

$$\begin{cases} x' = x + x_{\boldsymbol{t}} \\ y' = y + y_{\boldsymbol{t}} \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_{\boldsymbol{t}} \\ y_{\boldsymbol{t}} \end{bmatrix}$$

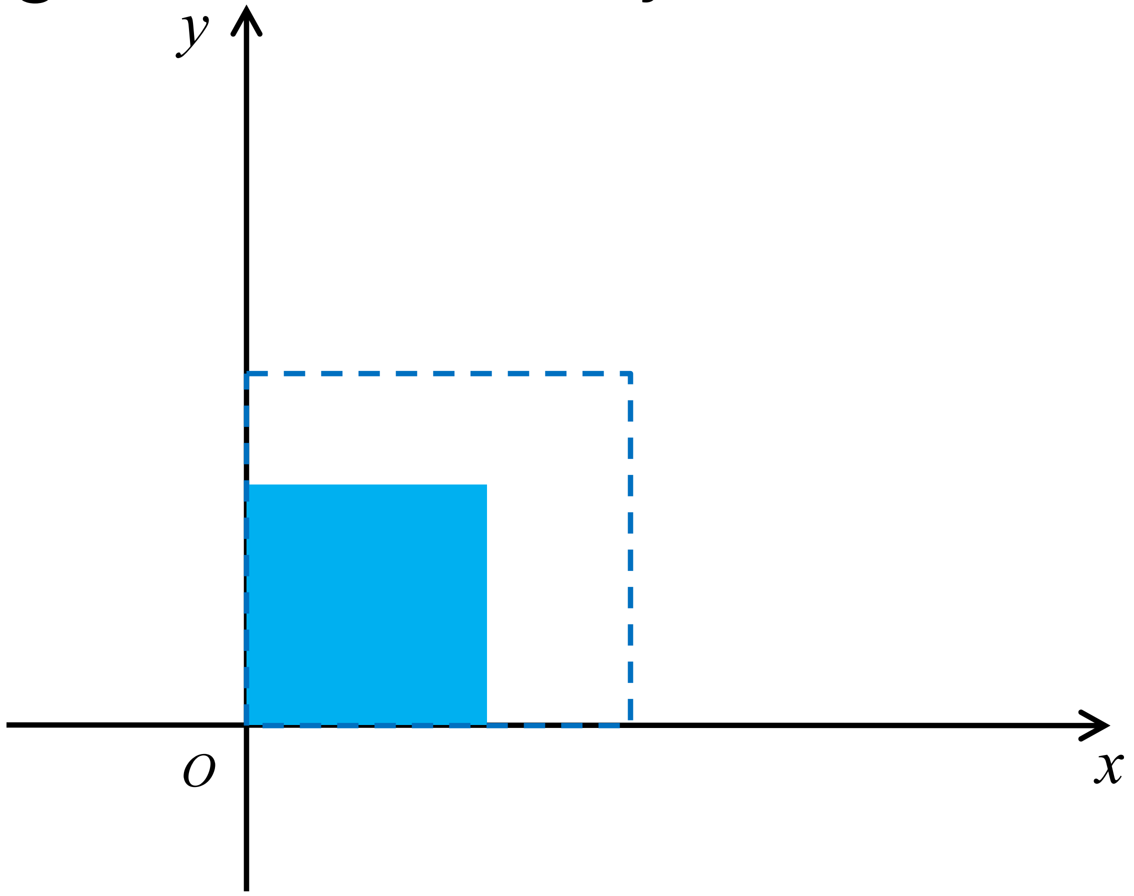
2D Scaling

- Change the size of object



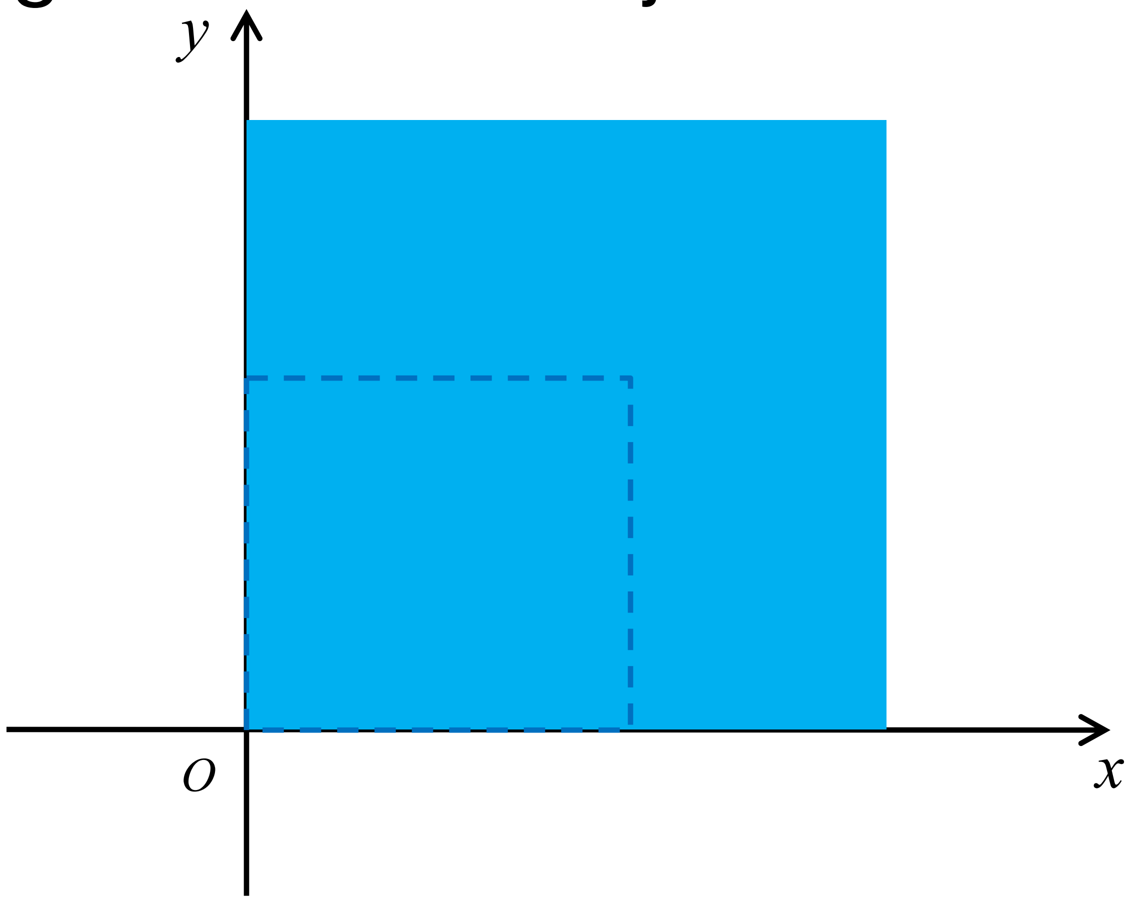
2D Scaling

- Change the size of object



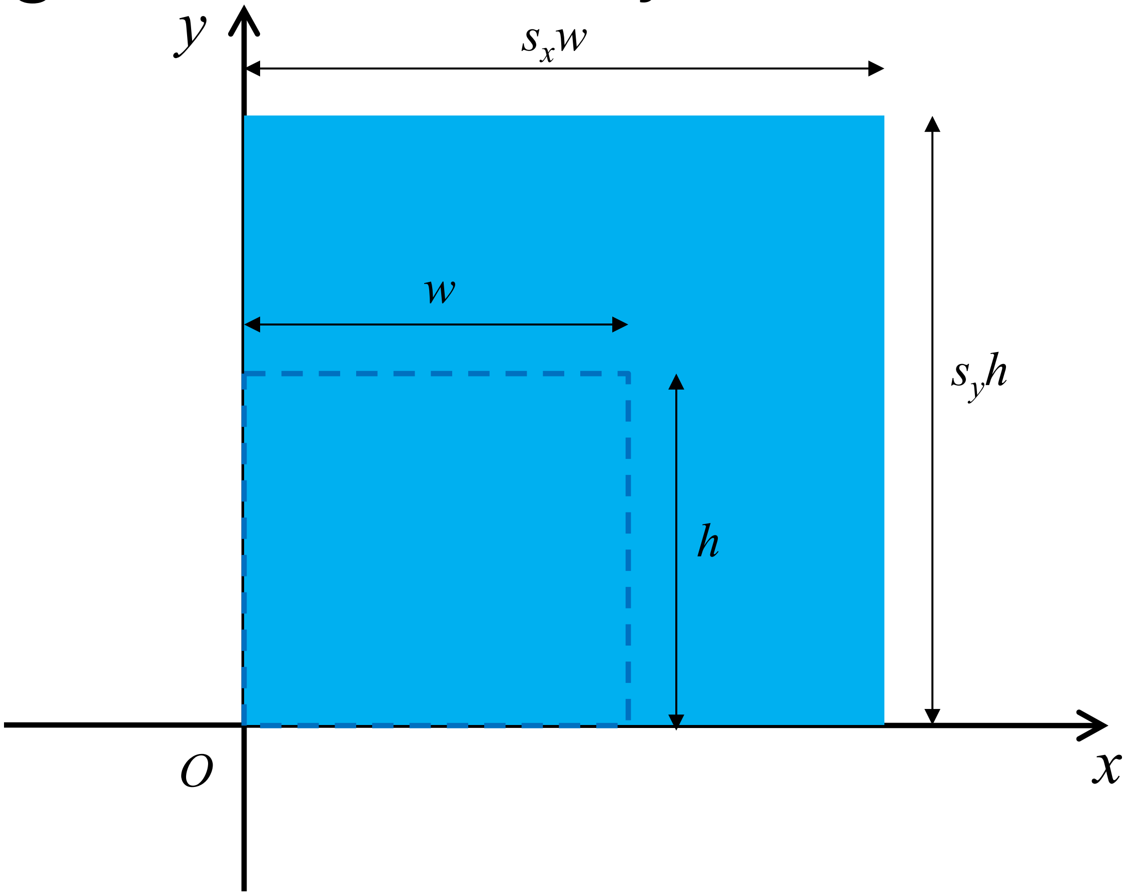
2D Scaling

- Change the size of object



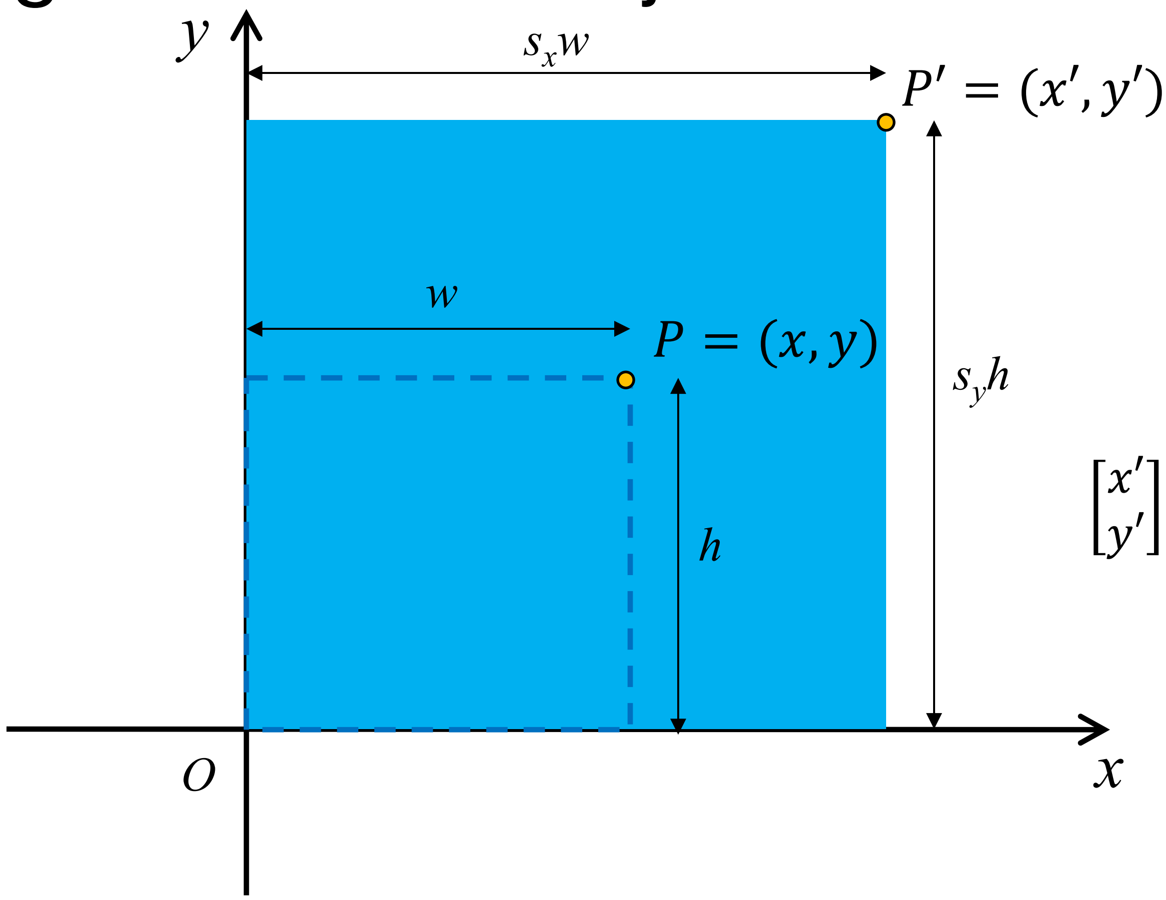
2D Scaling

- Change the size of object



2D Scaling

- Change the size of object



$$\begin{cases} x' = s_x x \\ y' = s_y y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Scaling Matrix S

$$P' = SP$$

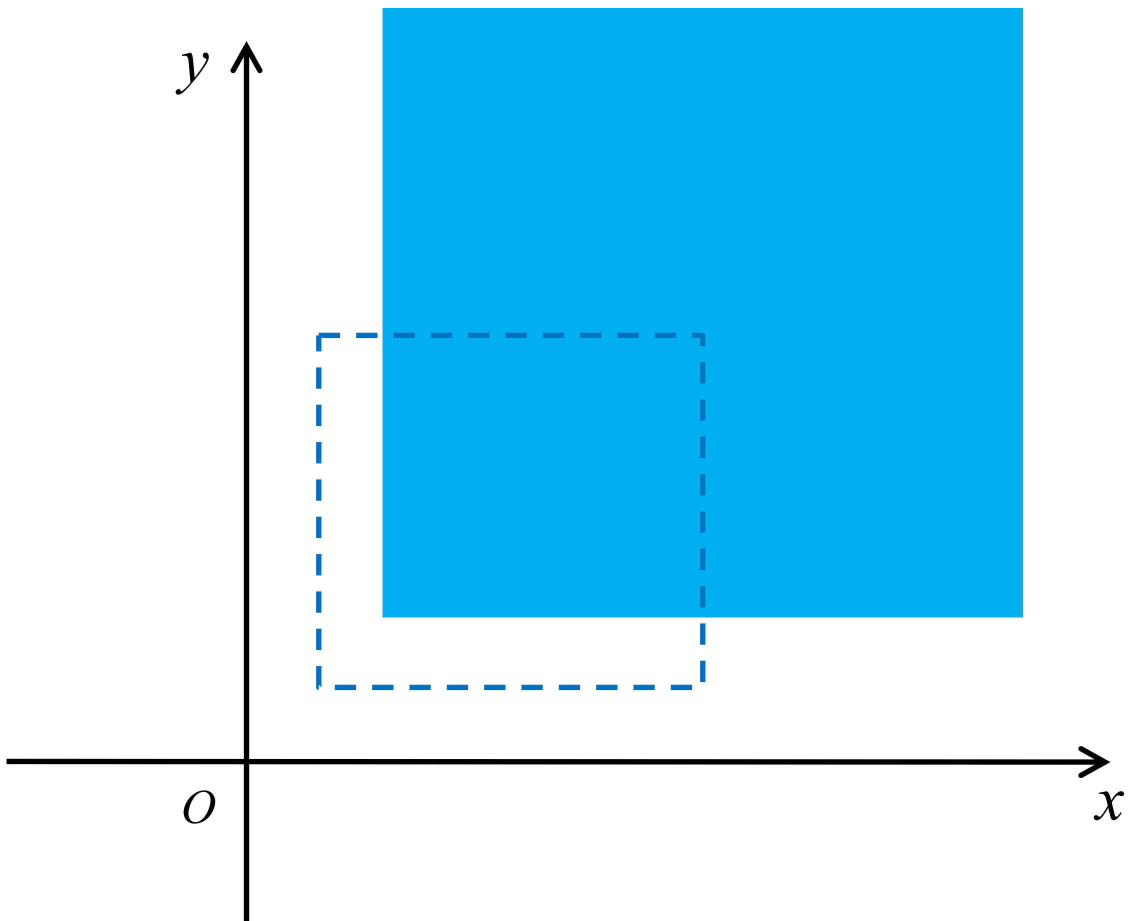
Properties

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

- Scaling factors (s_x or s_y) are *always greater* than zero
- Uniform scaling: $s_x = s_y$
  - Keep aspect ratio
- Differential scaling: $s_x \neq s_y$
  - Change aspect ratio
- Enlarge: scaling factor >1
- Shrink: scaling factor < 1

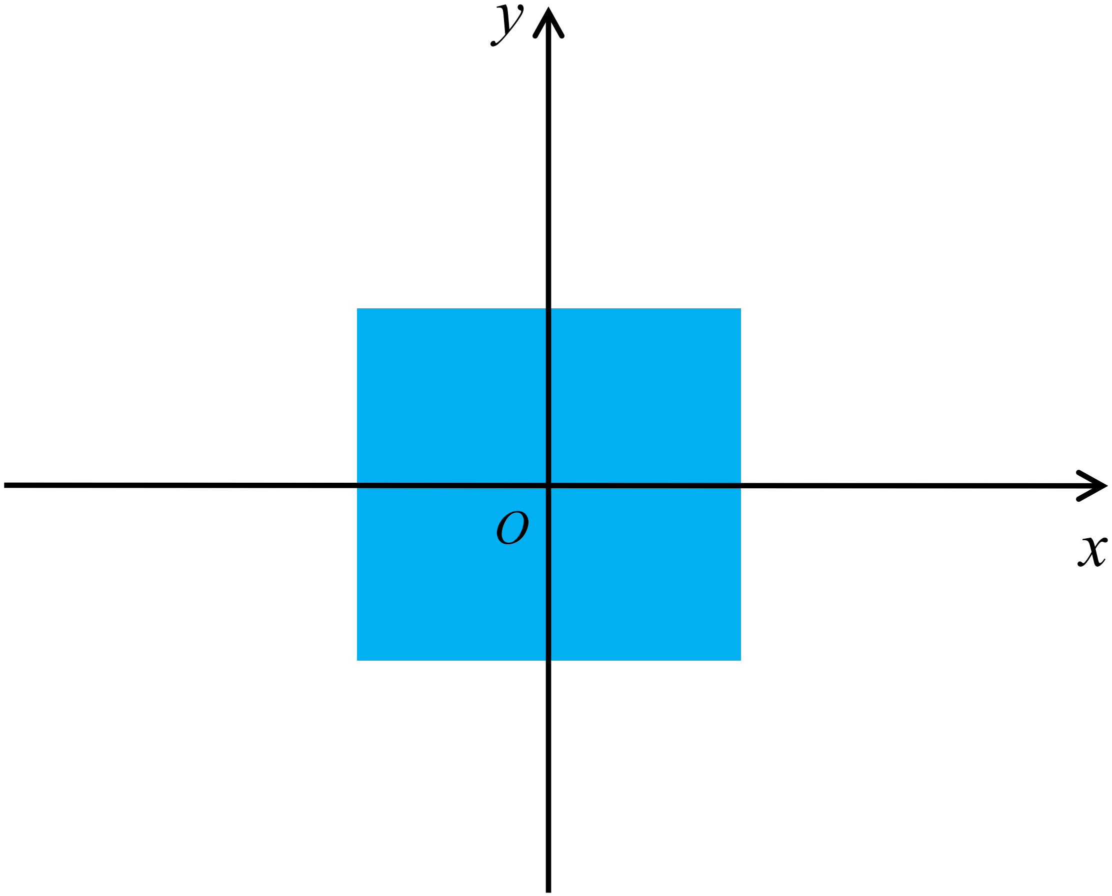
Properties

- The object is both *scaled* and *repositioned*



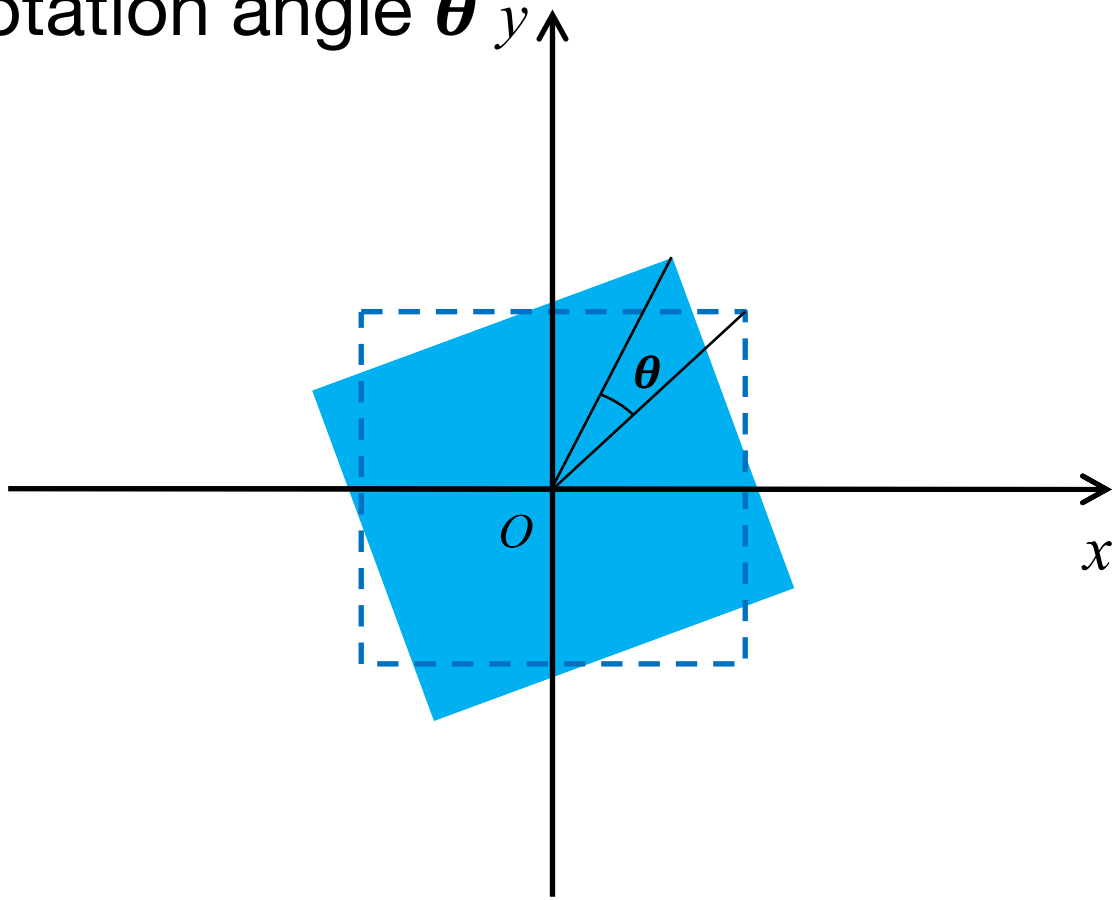
2D Rotation

- Change the orientation of object



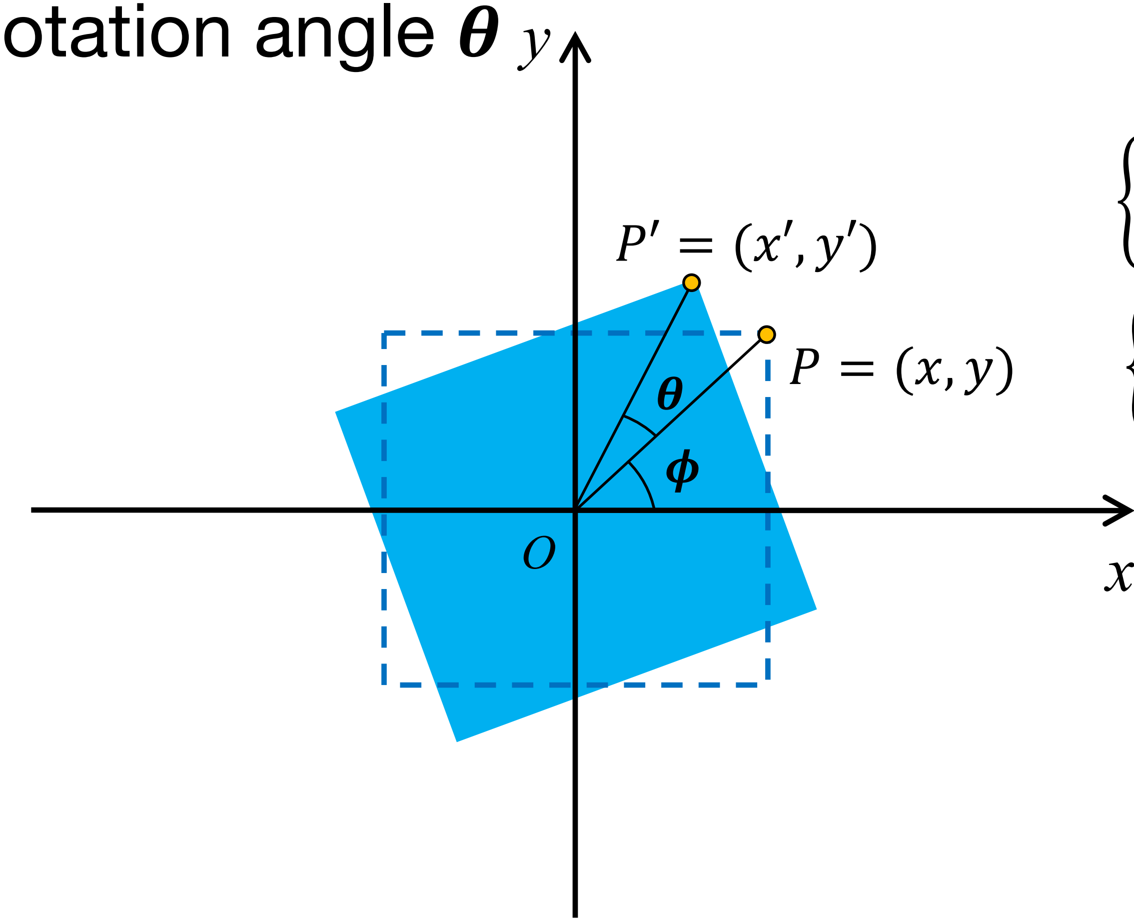
2D Rotation

- Change the orientation of object
  - Rotation angle $\boldsymbol{\theta}$



2D Rotation

- Change the orientation of object
  - Rotation angle $\boldsymbol{\theta}$

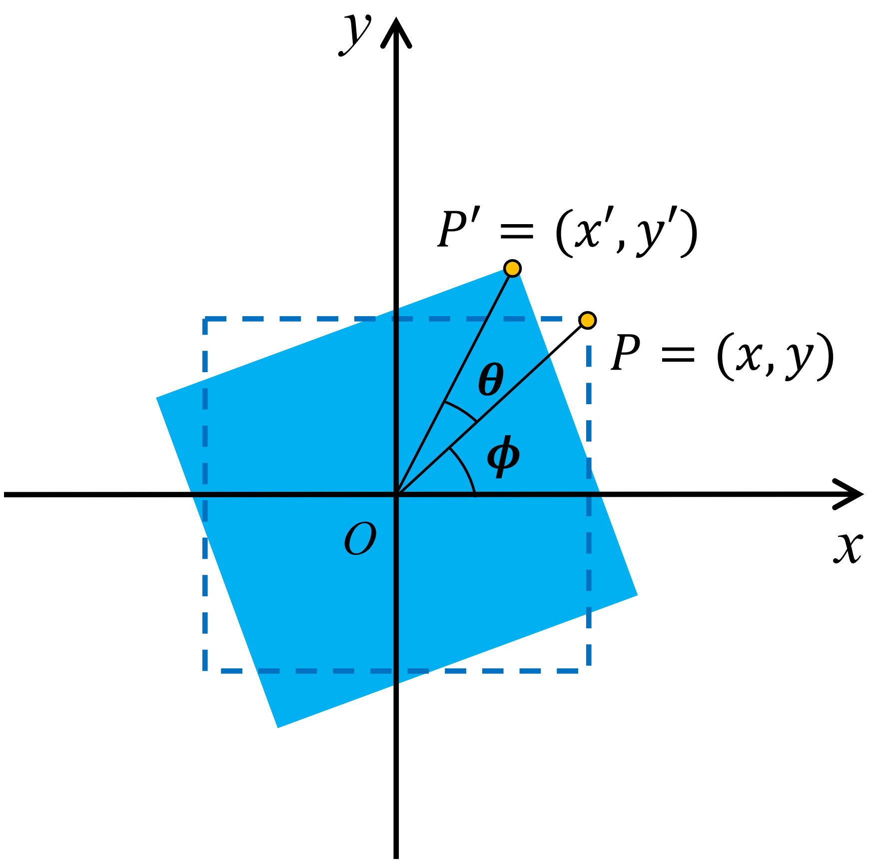


$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$\begin{cases} x' = r\cos(\theta + \phi) \\ y' = r\sin(\theta + \phi) \end{cases}$$

2D Rotation

- Derivation



$$\begin{cases} x' = r\cos(\theta + \phi) \\ y' = r\sin(\theta + \phi) \end{cases}$$

$$\begin{cases} x = r\cos\phi \\ y = r\sin\phi \end{cases}$$

$$\begin{cases} x' = r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ y' = r\cos\phi\sin\theta + r\sin\phi\cos\theta \end{cases}$$

$$\begin{cases} x' = x\cos\theta - y\sin\theta \\ y' = x\sin\theta + y\cos\theta \end{cases}$$

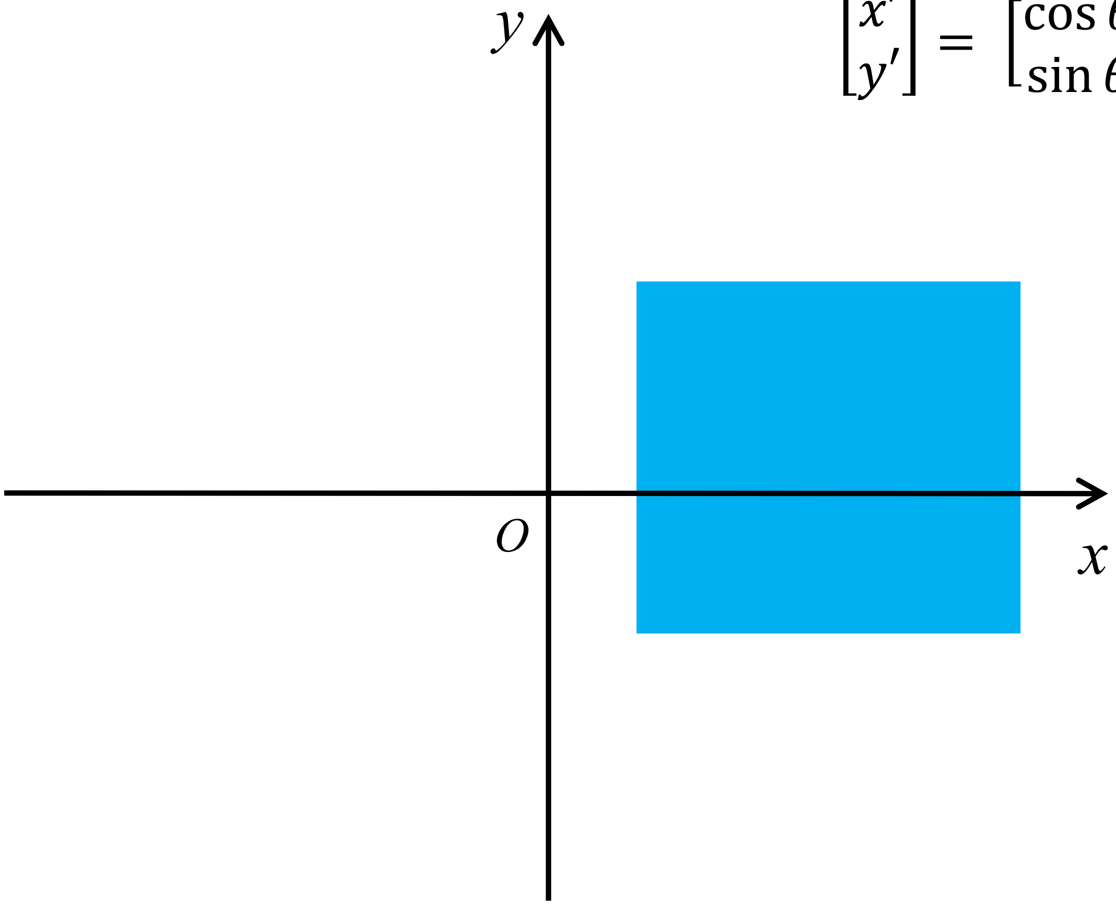
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Rotation Matrix R

$$P' = \mathrm{R}P$$

2D Rotation (Uncentric)

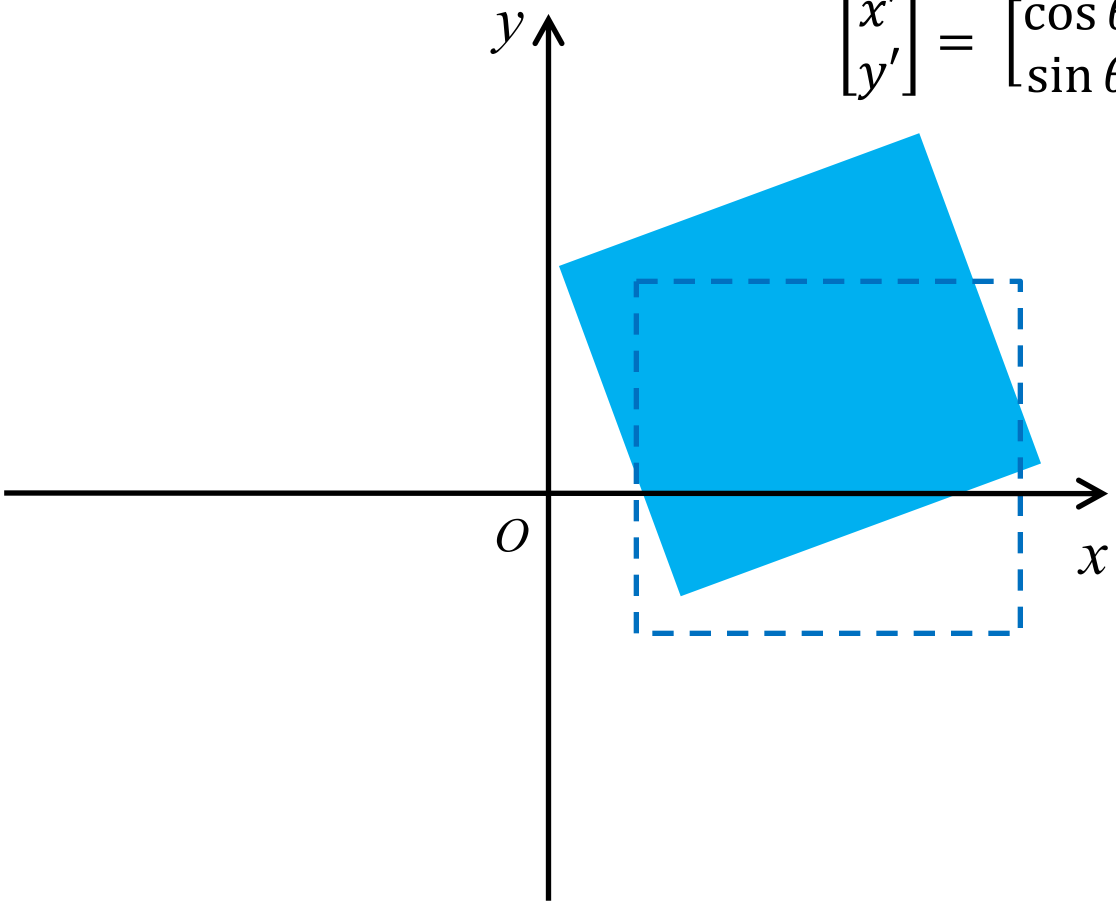
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



2D Rotation (Uncentric)

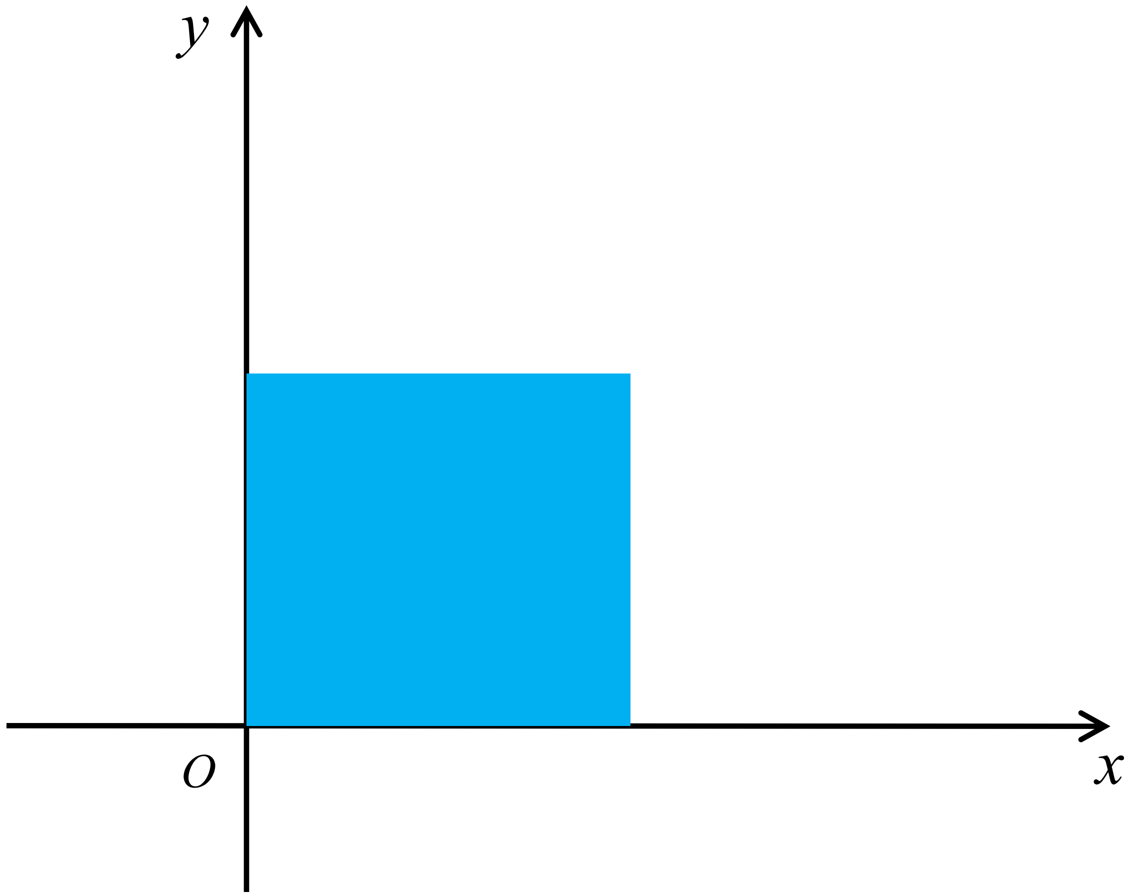
- Rotate about the origin

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



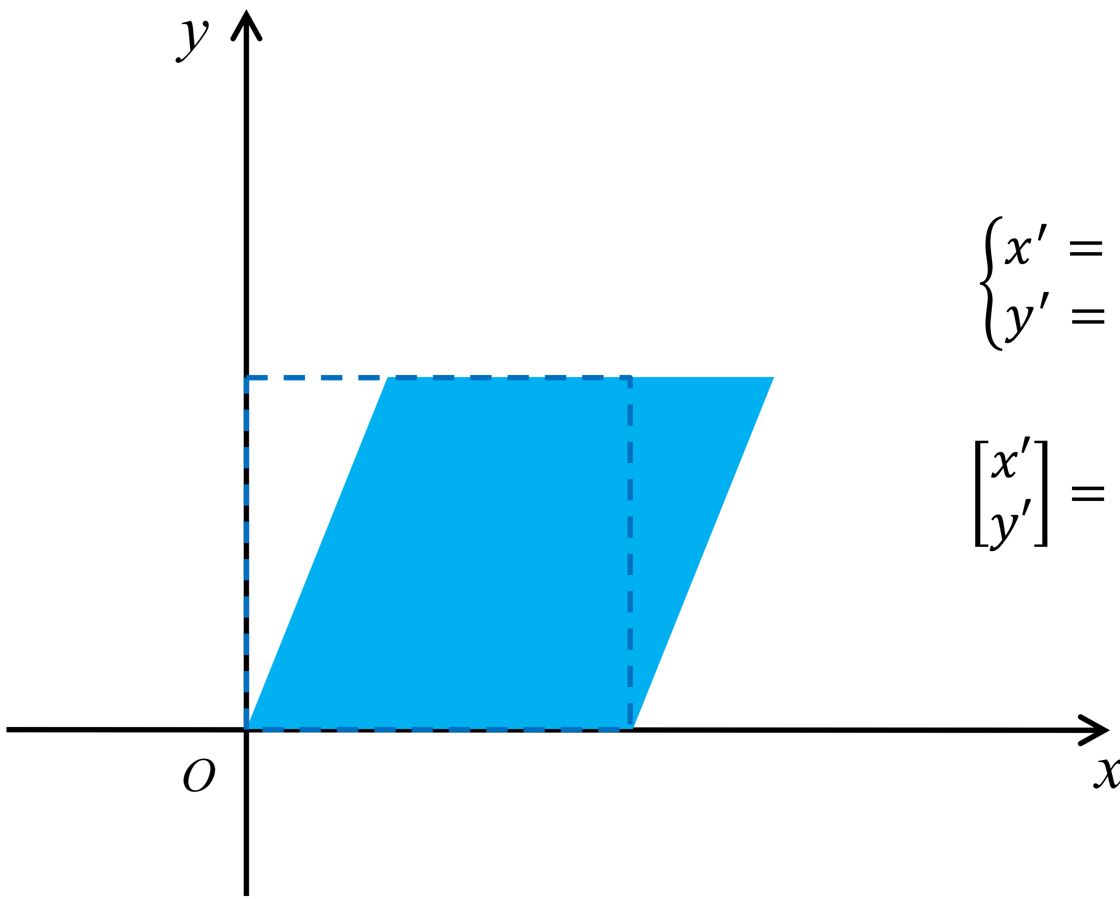
2D Shear

- Distort the shape of object



2D Shear (x-direction)

- Coordinates shift horizontally proportional to the vertical distance

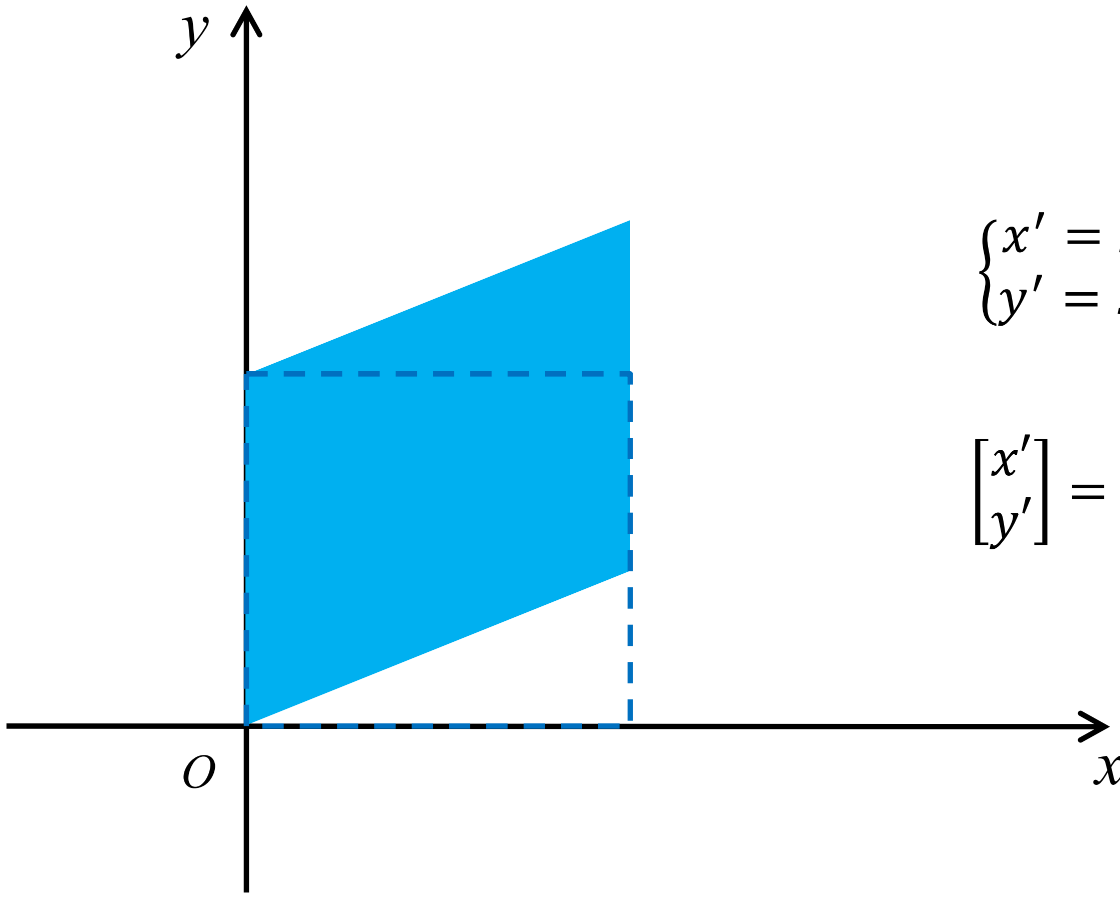


$$\begin{cases} x' = x + sh_x \cdot y \\ y' = y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & sh_x \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Shear (y-direction)

- Coordinates shift vertically proportional to the horizontal distance

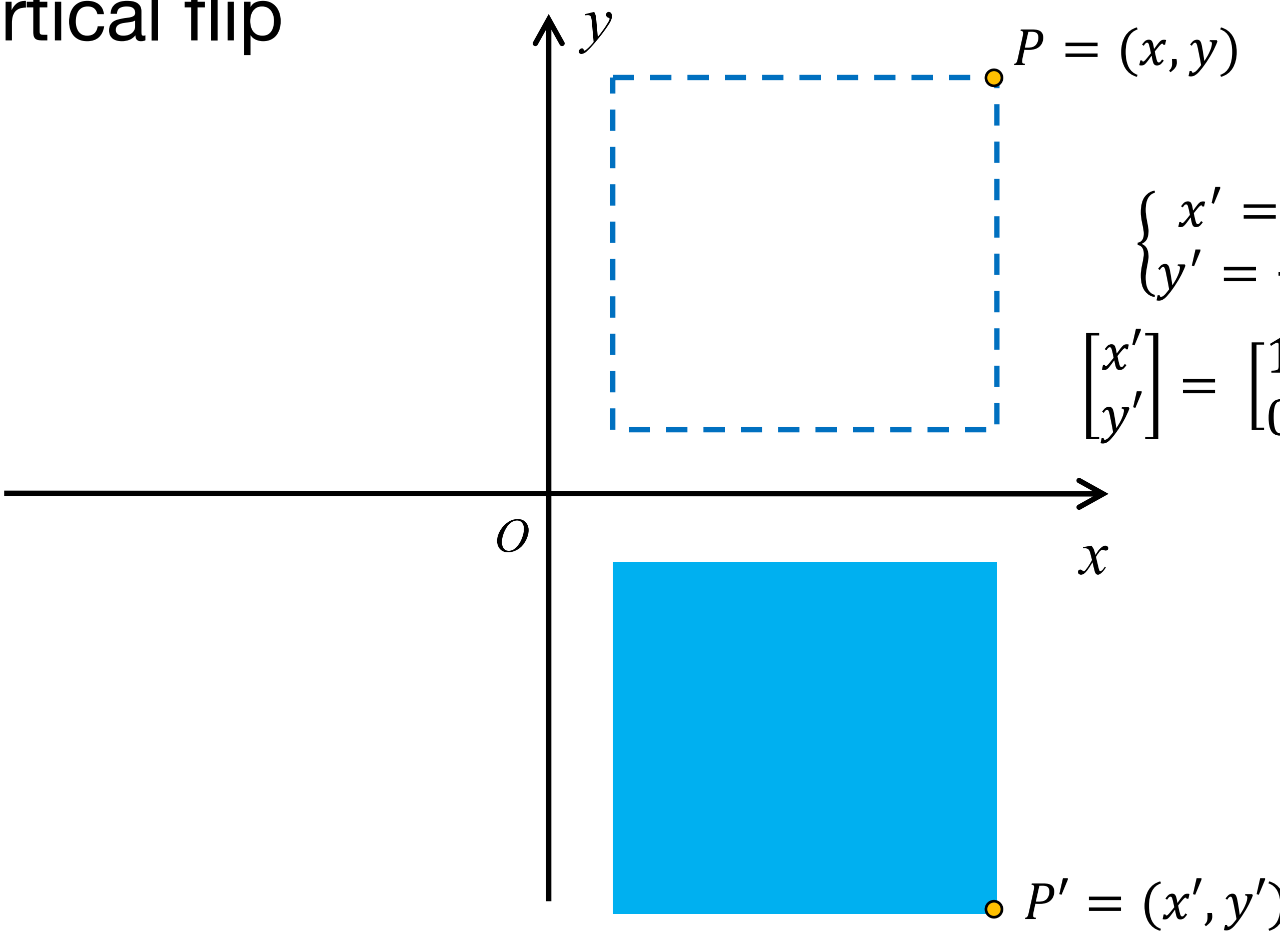


$$\begin{cases} x' = x \\ y' = sh_y \cdot x + y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ sh_y & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Reflection (x-axis)

- Flip about x-axis
  - vertical flip

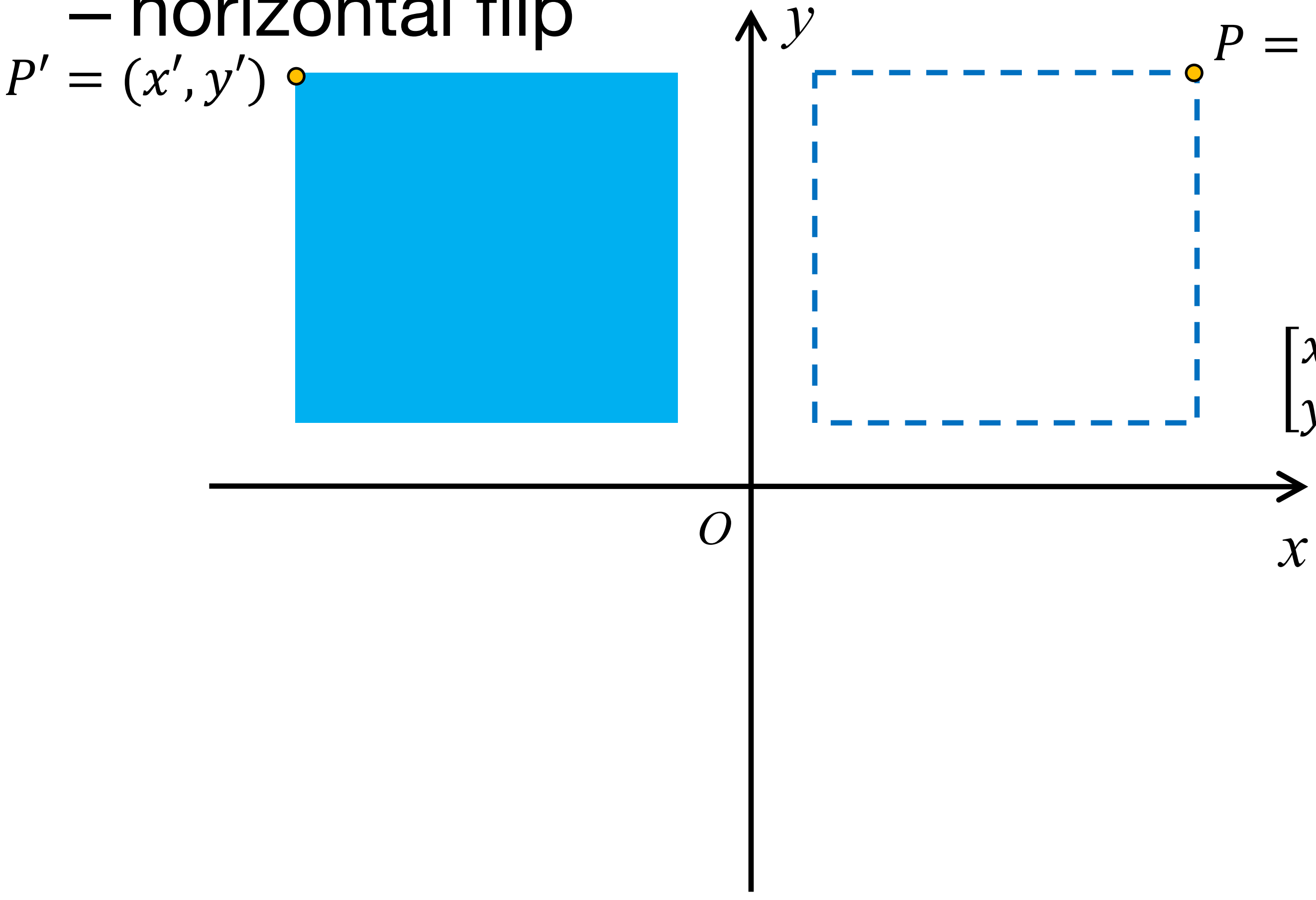


$$\begin{cases} x' = x \\ y' = -y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

2D Reflection (y-axis)

- Flip about y-axis
  - horizontal flip



$$\begin{cases} x' = -x \\ y' = y \end{cases}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Inverse Transformation

- Translation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} -x_{\boldsymbol{t}} \\ -y_{\boldsymbol{t}} \end{bmatrix} \qquad P = P' - \boldsymbol{t}$$

- Scaling

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1/s_x & 0 \\ 0 & 1/s_y \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad P = \mathrm{S}^{-1} P'$$

- Rotation

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad P = \mathrm{R}^{-1} P'$$

Inverse Transformation

- Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \qquad P' = \mathrm{R}P$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix} \qquad P = \mathrm{R}^{-1}P'$$

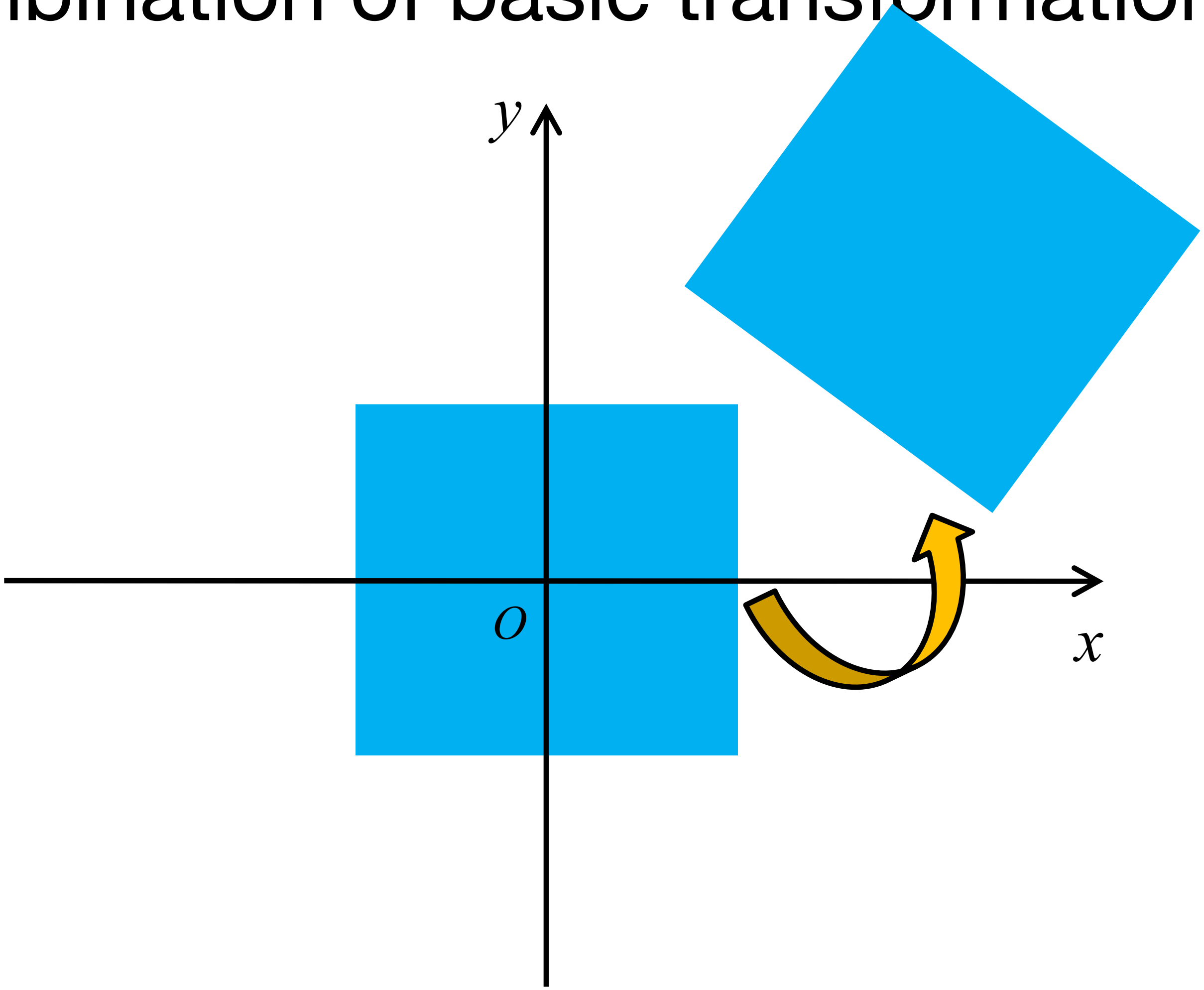
$$\mathrm{R}^{-1} = \mathrm{R}^{\mathrm{T}}$$

$$\mathrm{R}\mathrm{R}^{\mathrm{T}} = \mathrm{R}^{\mathrm{T}}\mathrm{R} = 1$$

Rotation matrix is orthogonal

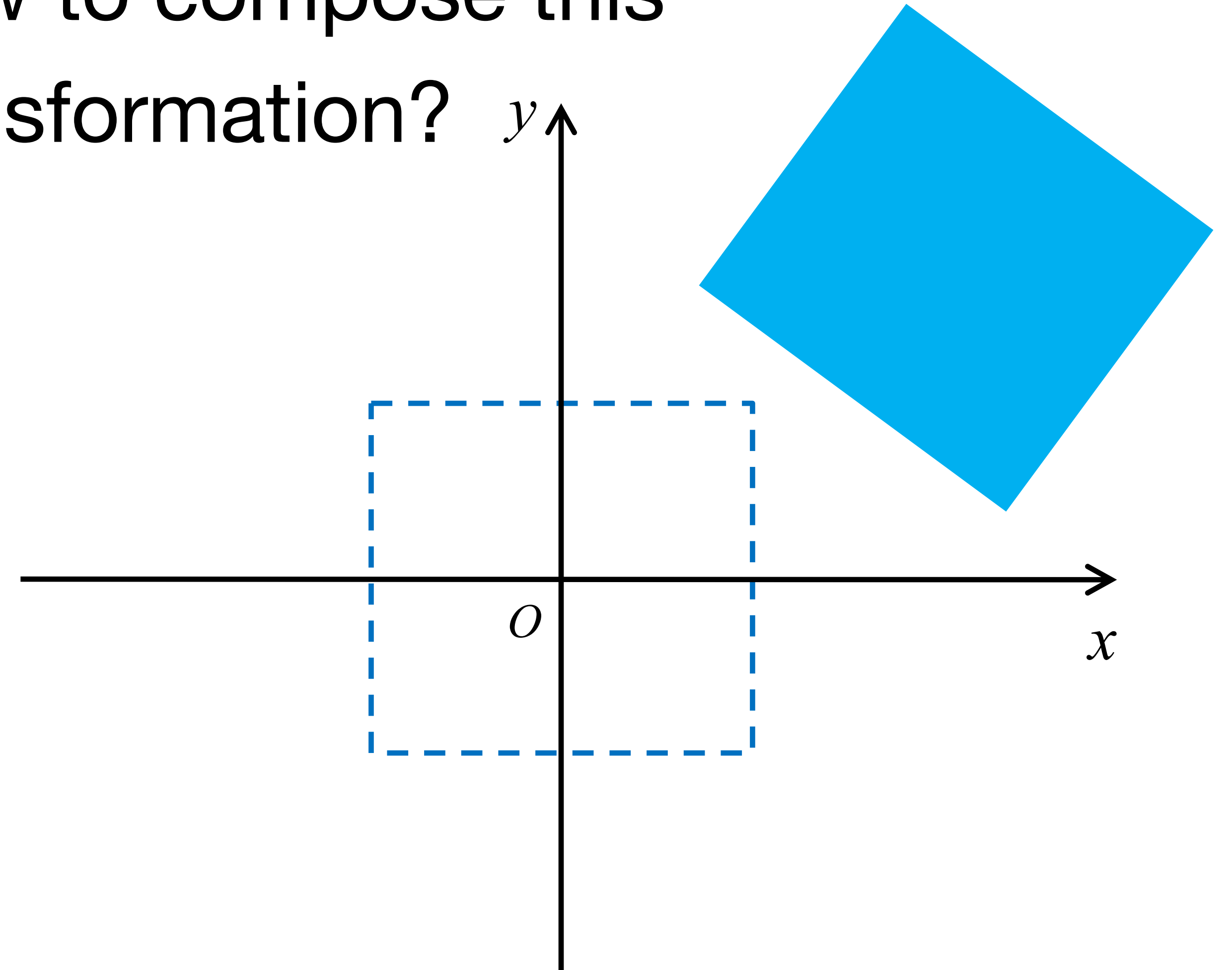
Composite Transformation

- Combination of basic transformations



Composite Transformation

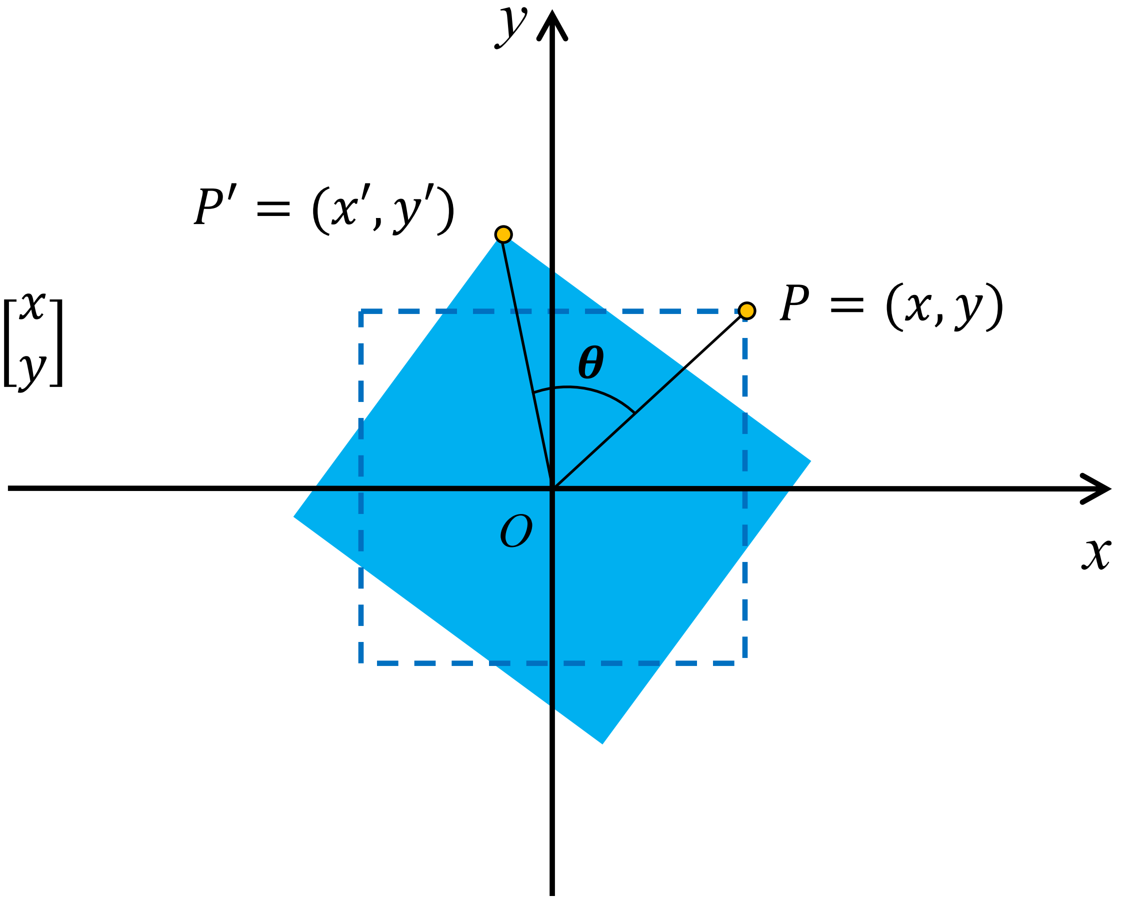
- How to compose this transformation?



Composite Transformation

Step 1: Rotate $\boldsymbol{\theta}$

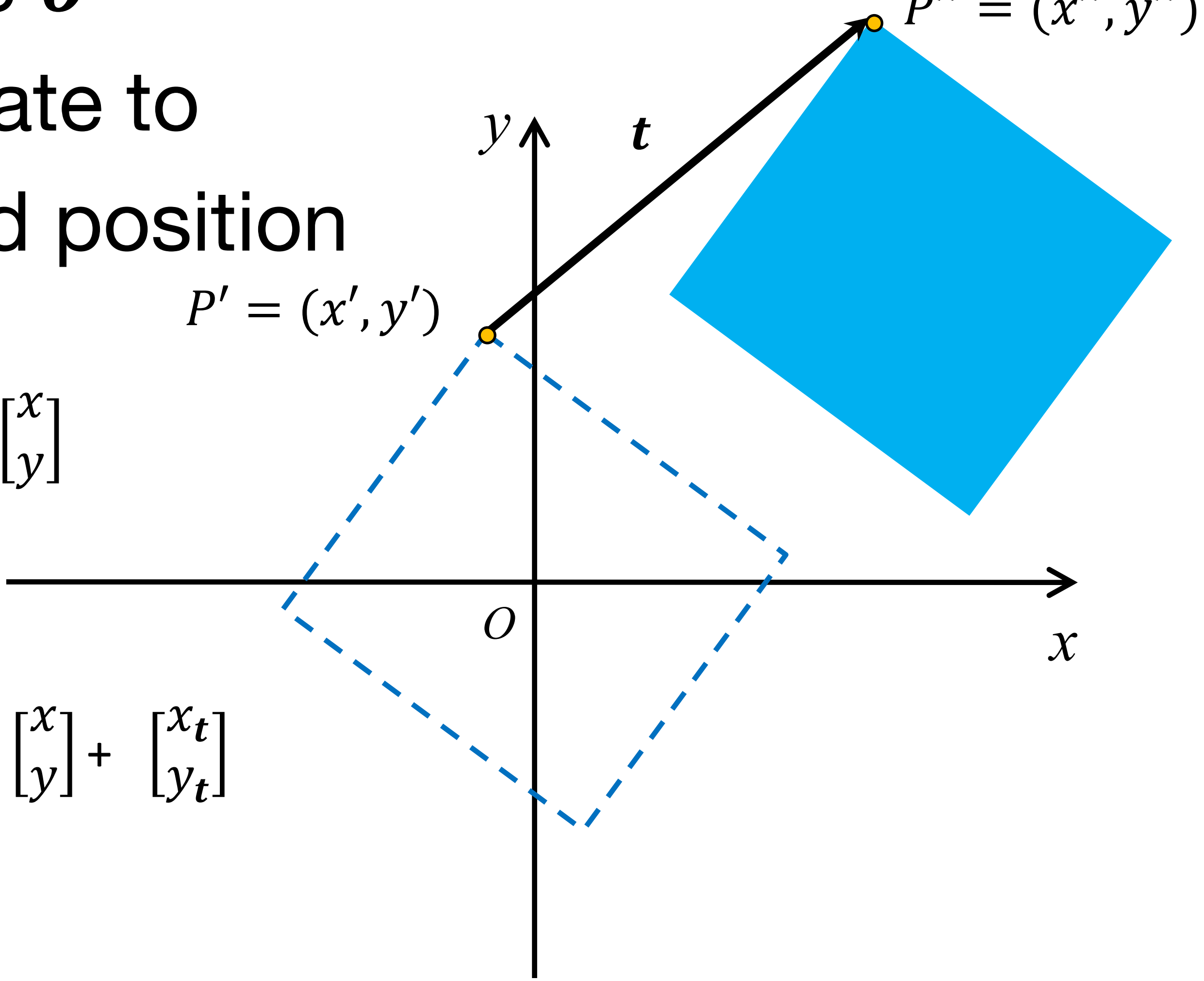
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



Composite Transformation

Step 1: Rotate $\boldsymbol{\theta}$

Step 2: Translate to desired position



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

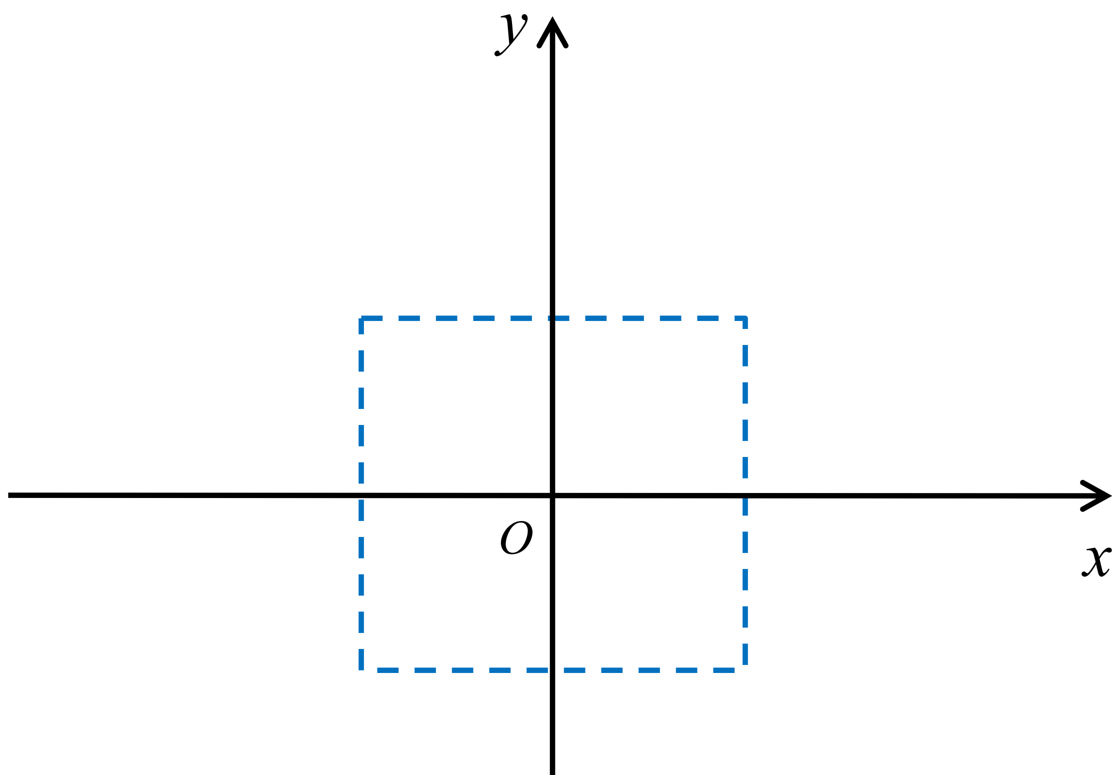
$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix} + \begin{bmatrix} x_{\boldsymbol{t}} \\ y_{\boldsymbol{t}} \end{bmatrix}$$

$$\begin{bmatrix} x'' \\ y'' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_{\boldsymbol{t}} \\ y_{\boldsymbol{t}} \end{bmatrix}$$

$$P'' = \mathrm{R}P + \boldsymbol{t}$$

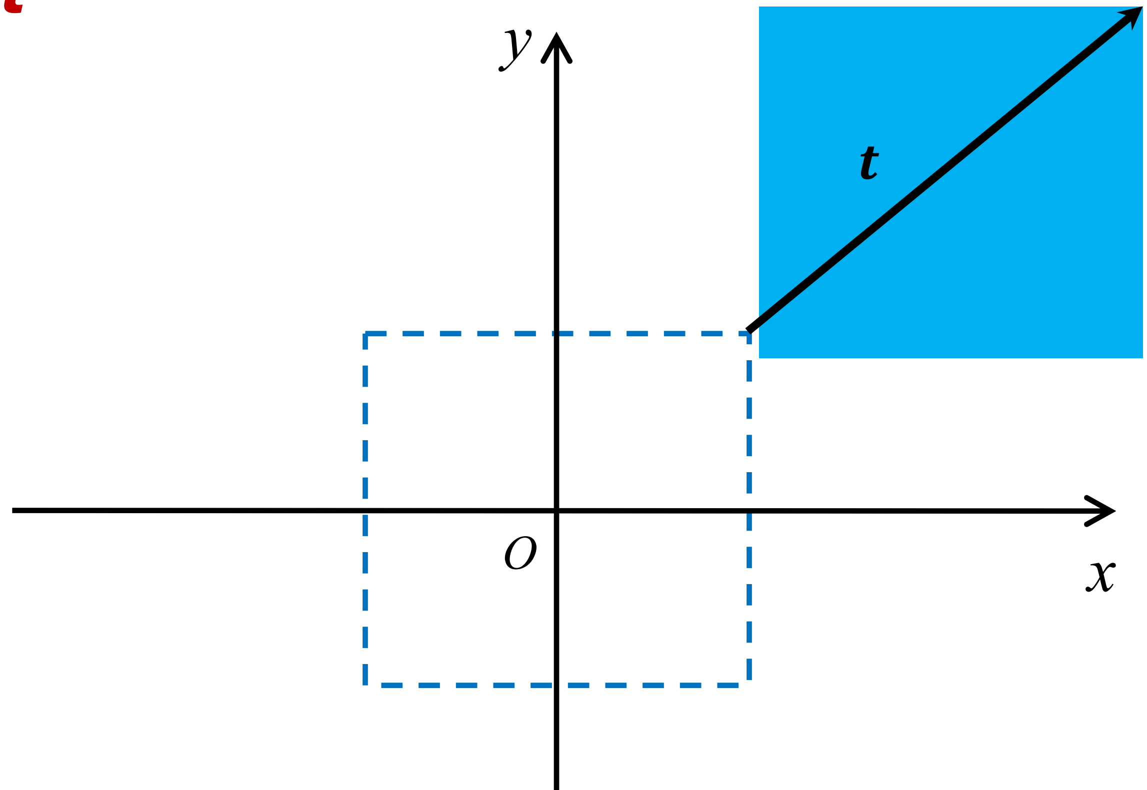
Composite Transformation

- Does order matter?
- Let's try changing the order:
  1. Translate $\boldsymbol{t}$
  2. Rotate $\boldsymbol{\theta}$



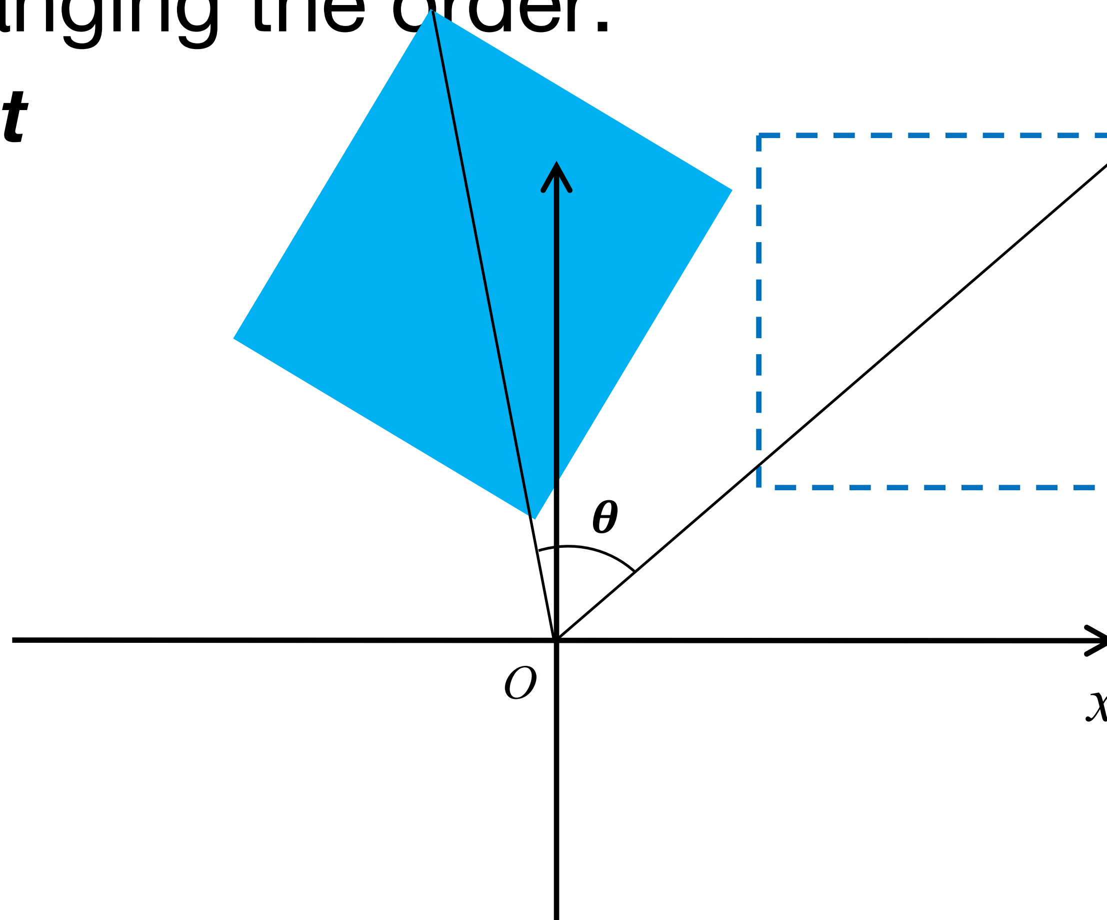
Composite Transformation

- Does order matter?
- Let's try changing the order:
    1. Translate $\boldsymbol{t}$
    2. Rotate $\boldsymbol{\theta}$



Composite Transformation

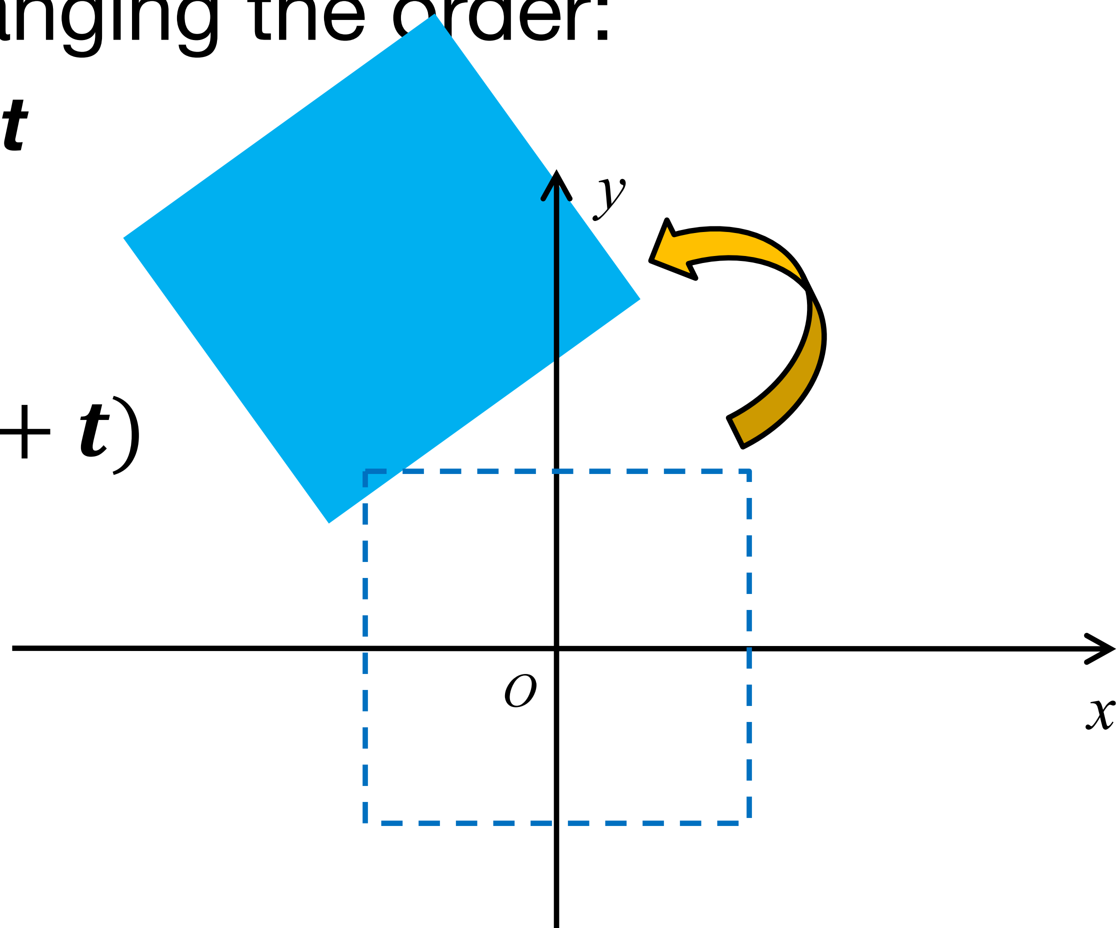
- Does order matter?
- Let's try changing the order:
  1. Translate ***t***
  2. Rotate ***θ***



Composite Transformation

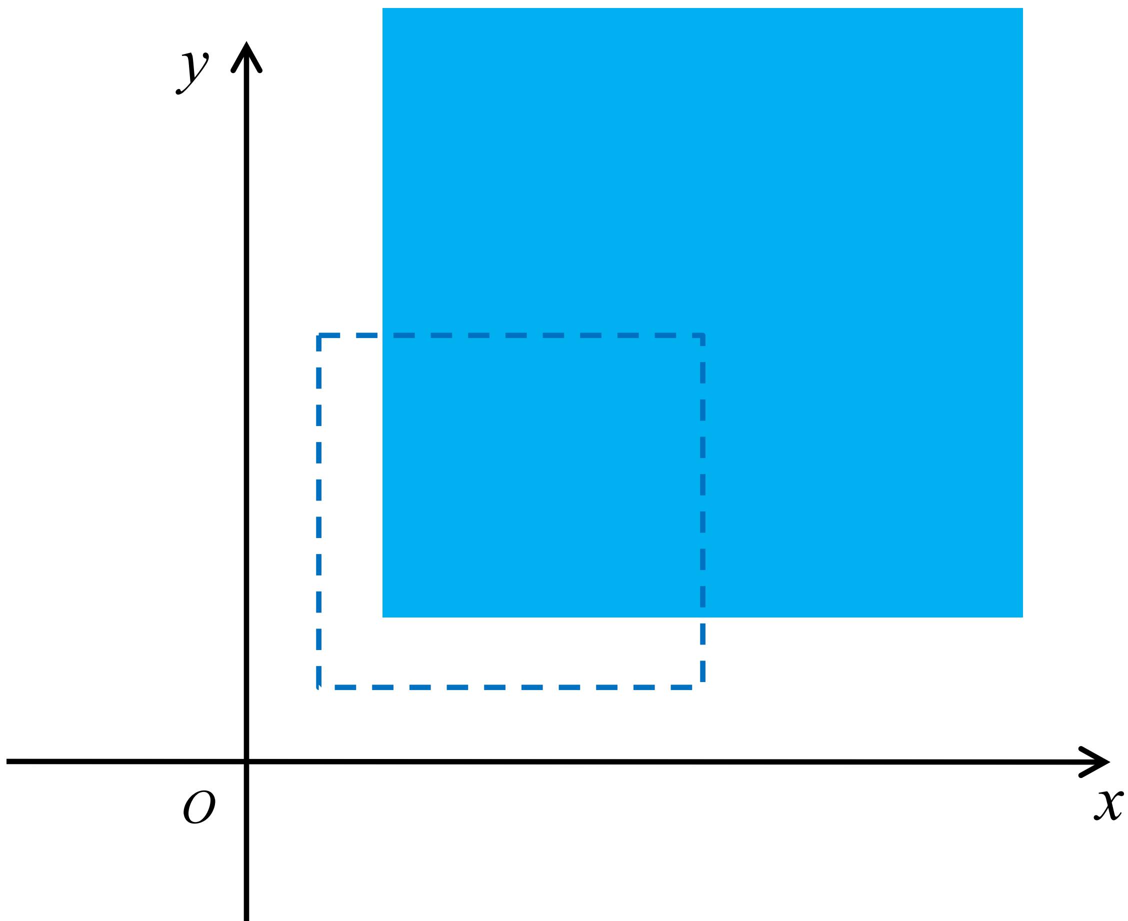
- Does order matter? *Yes!*
- Let's try changing the order:
  1. Translate $\boldsymbol{t}$
  2. Rotate $\boldsymbol{\theta}$

$$\mathrm{R}P + \boldsymbol{t} \neq \mathrm{R}(P + \boldsymbol{t})$$



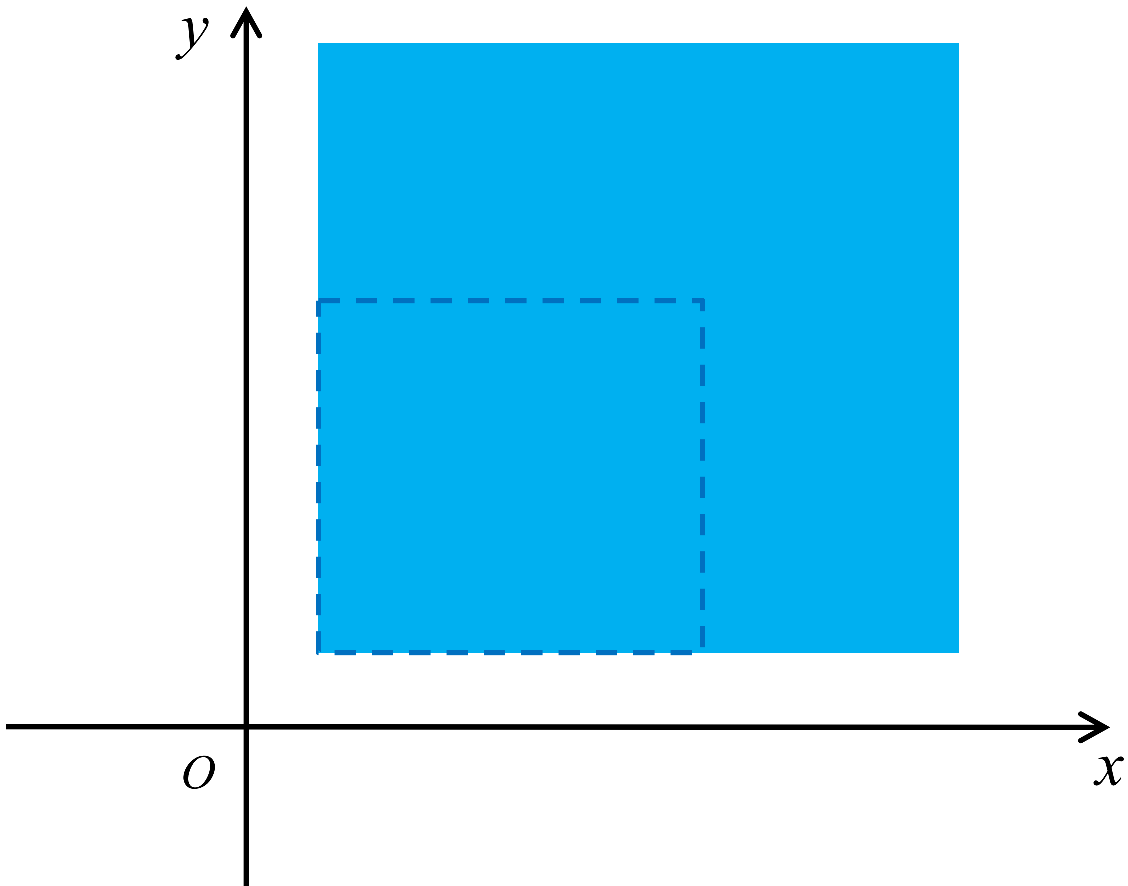
Scaling

- The object is both *scaled* and *repositioned*



Fixed Point Scaling

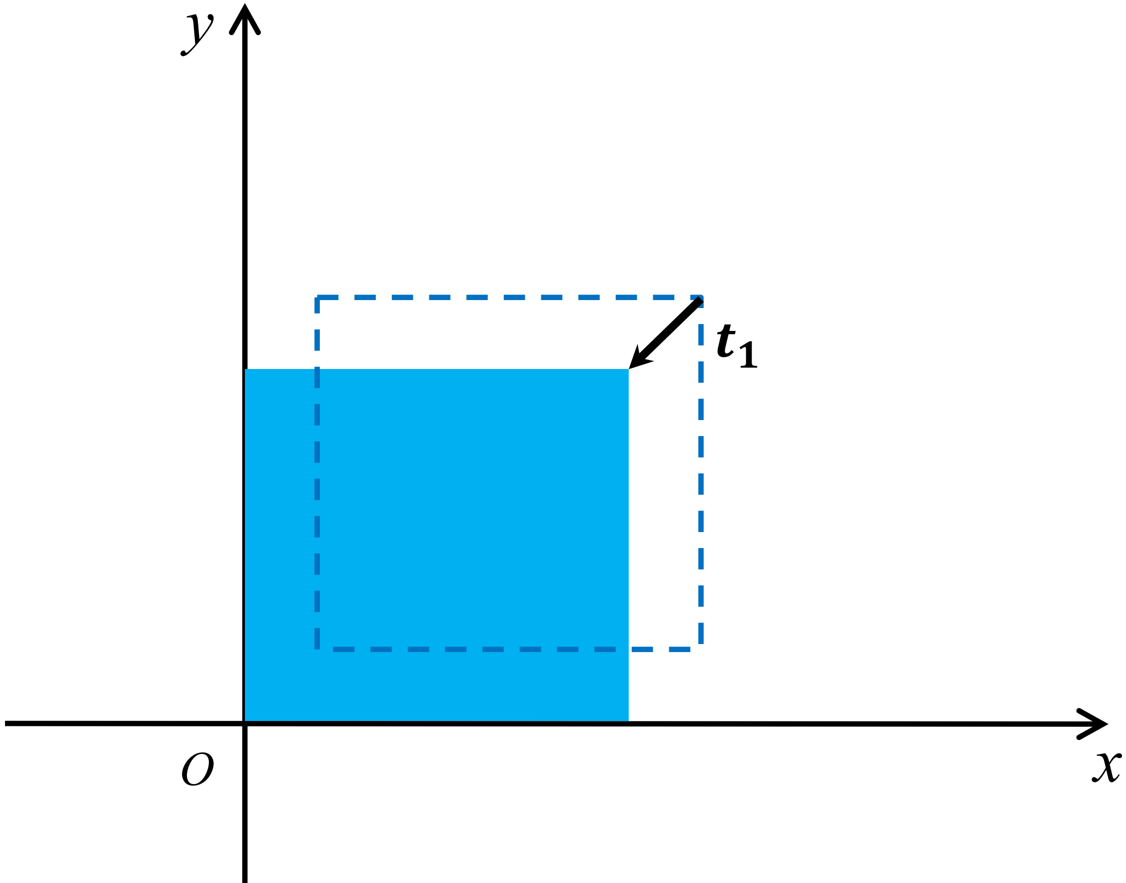
- Position relative to a reference point is fixed
  - Composite transformation



Fixed Point Scaling

Step1: Translate to origin

$$P' = P + \boldsymbol{t_1}$$



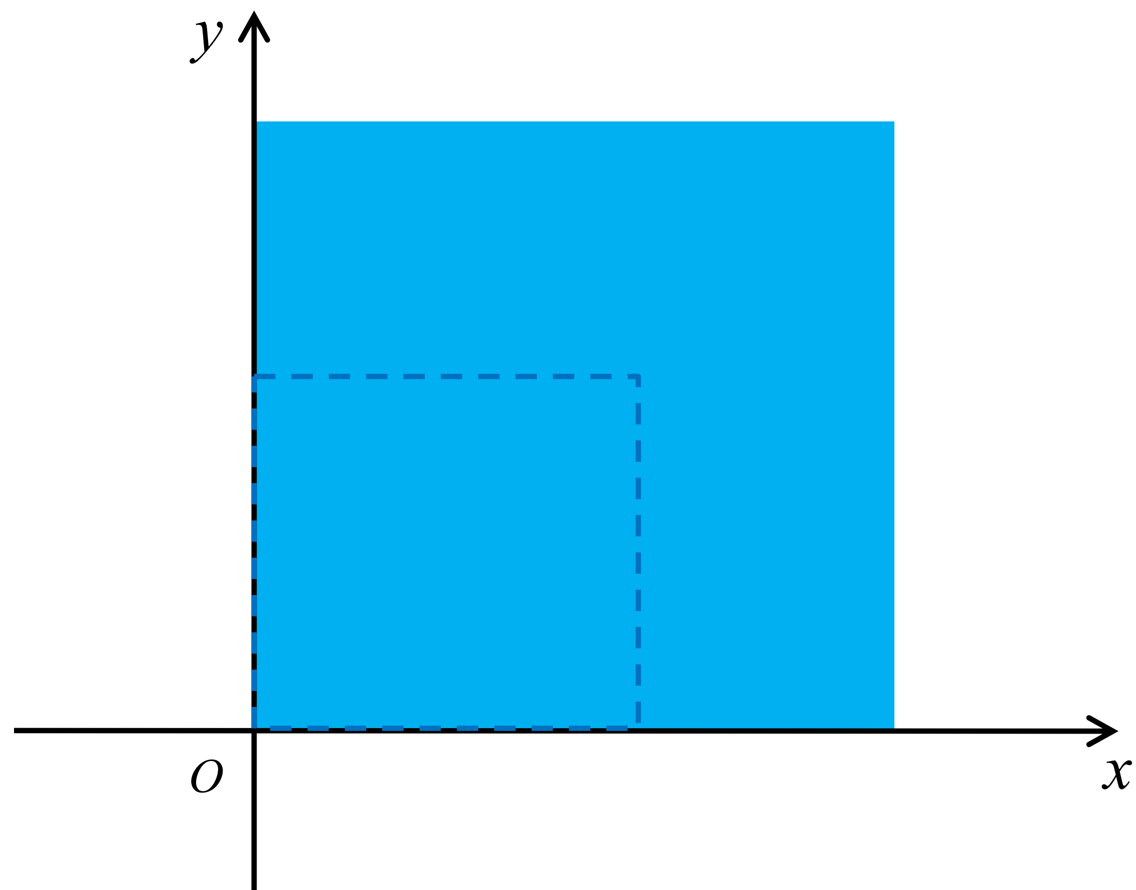
Fixed Point Scaling

Step 1: Translate to origin

Step 2: Apply scaling

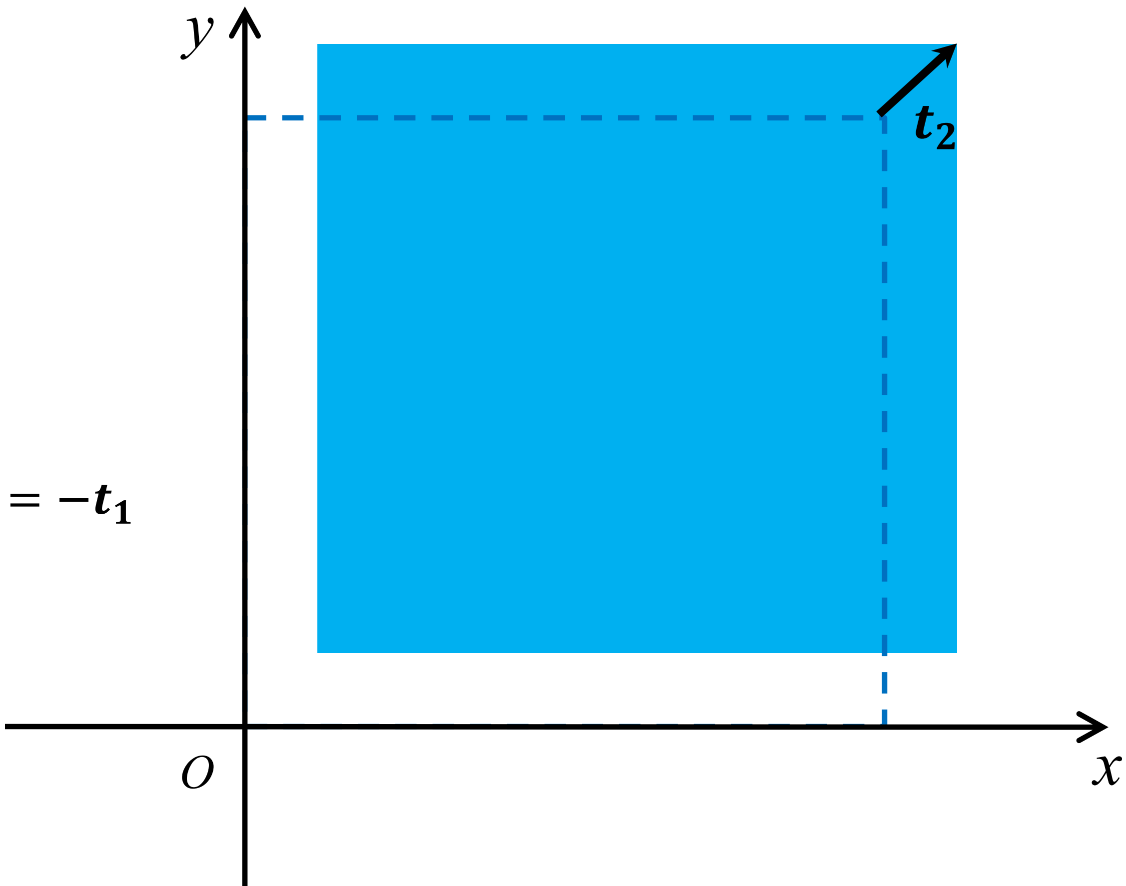
$P' = P + \boldsymbol{t_1}$

$P'' = SP'$



Fixed Point Scaling

Step 1: Translate to origin
Step 2: Apply scaling
Step 3: Translate back



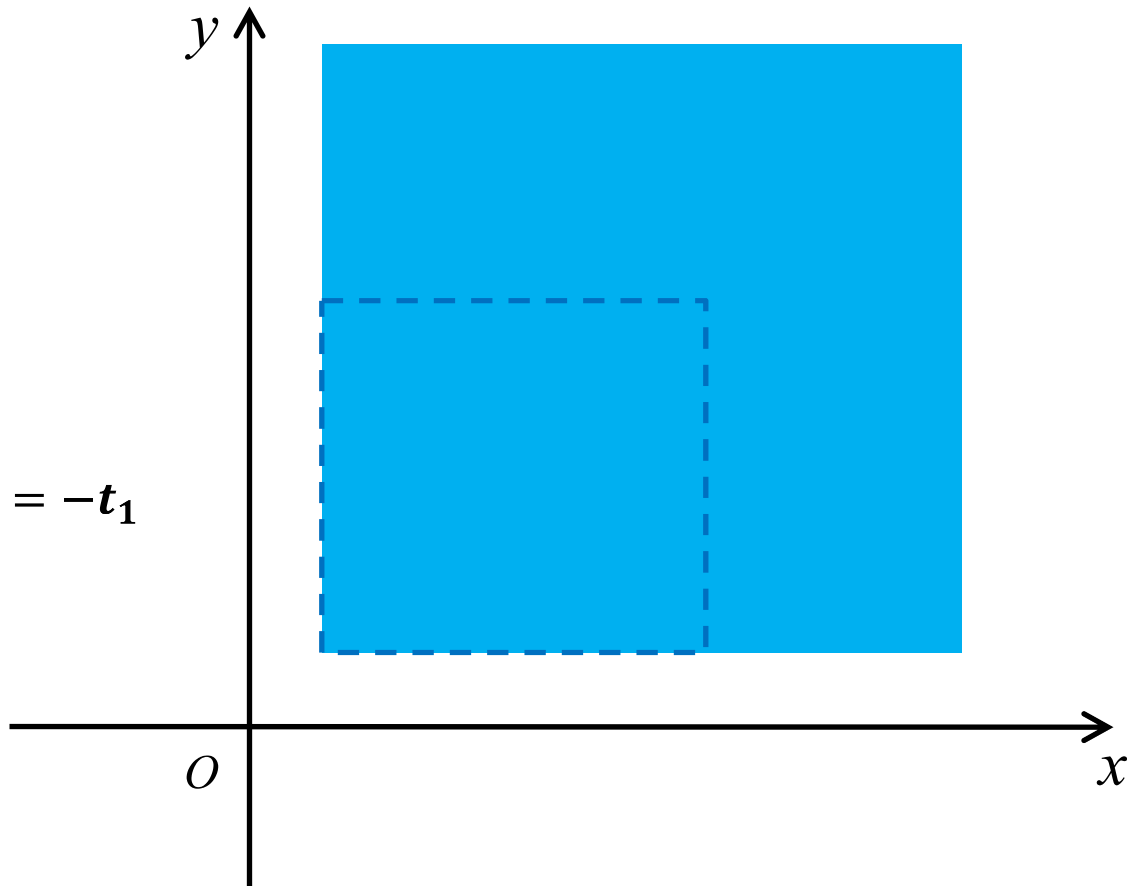
$$P' = P + t_1$$

$$P'' = SP'$$

$$P''' = P'' + t_2 \quad \text{where } t_2 = -t_1$$

Fixed Point Scaling

Step 1: Translate to origin
Step 2: Apply scaling
Step 3: Translate back



$P' = P + \boldsymbol{t_1}$

$P'' = SP'$

$P''' = P'' + \boldsymbol{t_2}$ where $\boldsymbol{t_2} = -\boldsymbol{t_1}$

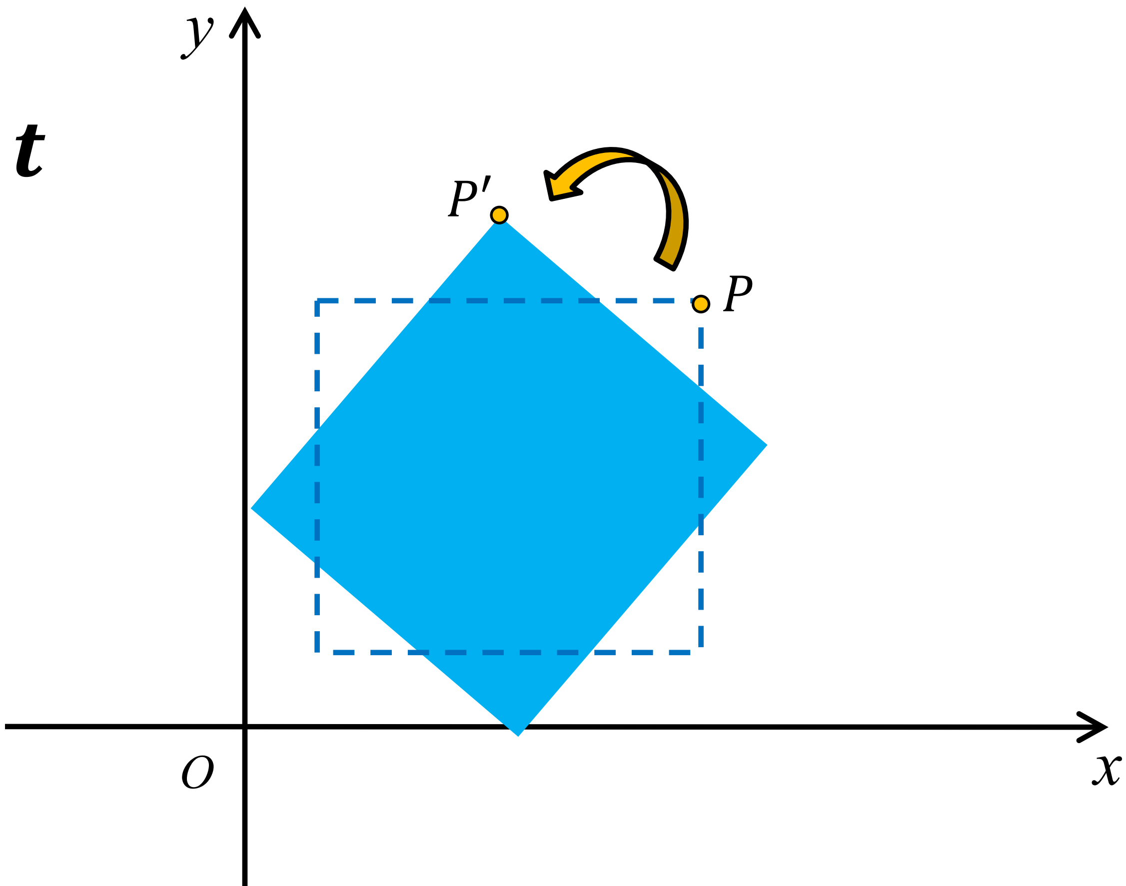
$P''' = S(P - \boldsymbol{t}) + \boldsymbol{t}$

where $\boldsymbol{t} = \boldsymbol{t_2} = -\boldsymbol{t_1}$

Arbitrary Rotation

Step 1: Translate to origin
Step 2: Apply rotation
Step 3: Translate back

$$P' = R(P - \boldsymbol{t}) + \boldsymbol{t}$$



Composite Transformations

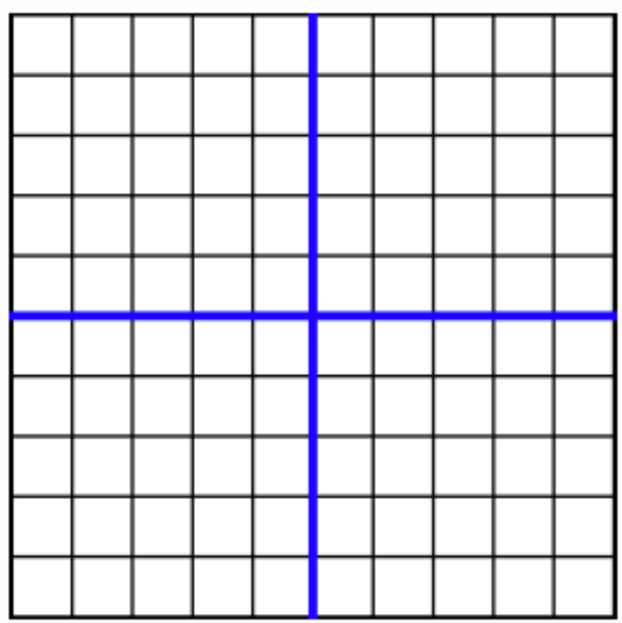
- A few examples:

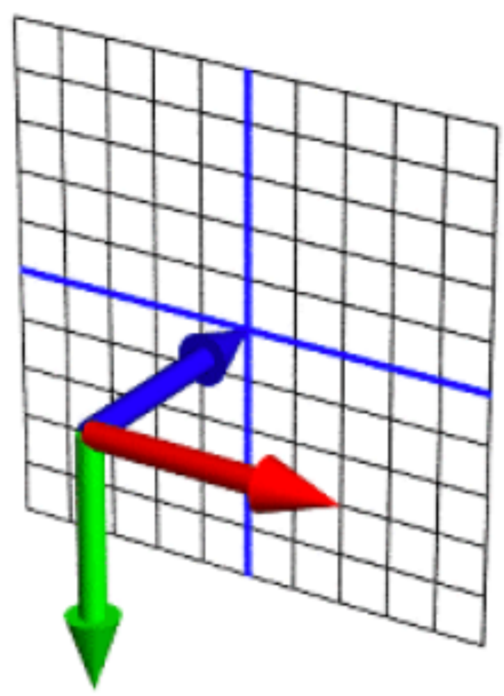
  $$\mathrm{R}P + \boldsymbol{t},\ \mathrm{R}(P + \boldsymbol{t}),\ S(P - \boldsymbol{t}) + \boldsymbol{t},\ R(P - \boldsymbol{t}) + \boldsymbol{t}$$

- Combinations of matrix addition (translation) and multiplication (rotation, scaling)
- Isn't it inconvenient?
  - Translation and rotation must be considered separately
  - Inversion involves multiple steps
- Any unified form for all transformations?

Subspace in 3D

- Solution: pick a subspace within 3D as our 2D coordinate plane





Homogeneous Coordinate

- WLOG, assume that our 2D subspace lies on the z=1 plane, such that our point coordinate changes to

$$[x, y]^{\mathrm{T}} \;\rightarrow\; [x, y, 1]^{\mathrm{T}}$$

- We call this subspace the Homogeneous Coordinate

Homogeneous Coordinate

- Combination of Rotation & Translation

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & x_t \\ \sin\theta & \cos\theta & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

- What is the order of transformation?
  - First rotate, then translation
- Old form: $\mathrm{R}P + \boldsymbol{t}$

Homogeneous Coordinate

- Translation only?

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & x_t \\ 0 & 1 & y_t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Rotation angle $\boldsymbol{\theta} = 0$

Homogeneous Coordinate

- Rotation, Scaling, Shear & Reflection?
  - Take rotation for example

Insert your 2 x 2 matrix

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Translation vector = **0**

Homogeneous Coordinate

- Now all transformations are expressed in matrix multiplication
- Composite transformation

$$\mathrm{X}' = \mathrm{M}_1\mathrm{M}_2\mathrm{M}_3\mathrm{X}$$

- Order matters!

$$\mathrm{M}_1\mathrm{M}_2\mathrm{X} \neq \mathrm{M}_2\mathrm{M}_1\mathrm{X}$$

- Inverse transformation

$$\mathrm{X} = M_3^{-1}M_2^{-1}M_1^{-1}\mathrm{X}'$$

Next Time …

- 3D Transformations
- Math problem set is due on Thursday!
- Turn in in-class
- No late days